GROUP-INVARIANT CR MAPPINGS

BY

DUSTY EDWARD GRUNDMEIER

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2011

Urbana, Illinois

Doctoral Committee:

Professor Jeremy Tyson, Chair
Professor John D'Angelo, Director of Research
Professor Christopher Leininger
Adjunct Professor Jiří Lebl

Abstract

We consider group-invariant CR mappings from spheres to hyperquadrics. Given a finite subgroup $\Gamma \subset U(n)$, a construction of D'Angelo and Lichtblau yields a target hyperquadric $Q(\Gamma)$ and a canonical non-constant CR map $h_\Gamma : S^{2n-1}/\Gamma \to Q(\Gamma)$. For every $\Gamma \subset SU(2)$, we determine this hyperquadric $Q(\Gamma)$, that is, the numbers of positive and negative eigenvalues in its defining equation. For families of cyclic and dihedral subgroups of $U(2)$, we study these numbers asymptotically as the order of the group tends to infinity. Next we study number-theoretic and combinatorial aspects of h_Γ for cyclic $\Gamma \subset U(2)$. In particular, we show that the mappings h_Γ associated to the lens spaces $L(p,q)$ satisfy a linear recurrence relation of order $2^q - 1$ and no smaller. We also give explicit but complicated formulas for the coefficients. Finally, we explore connections with representation theory and invariant theory.

To my parents.

Acknowledgments

First, I'd like to give my sincere thanks to my advisor, John D'Angelo. His guidance and encouragement have been crucial to my success in graduate school. He introduced me to a beautiful area of mathematics, and I will always be grateful for that. He also carefully read several drafts of this thesis and helped improve the exposition. I would also like to thank Jiří Lebl for many mathematical discussions about the ideas in this thesis and for suggesting new problems to consider. I'd also like to thank Jeremy Tyson and Christopher Leininger for agreeing to be on my committee and for reading this thesis.

I acknowledge support from NSF grant DMS 08-38434 "EMSW21-MCTP: Research Experience for Graduate Students" and from NSF grant DMS 07-53978 of John D'Angelo. I would also like to thank the organizers of the Park City Math Institute and the American Institute of Mathematics for giving me opportunities to give short talks on the results of this thesis. The main results in chapters 2 and 3 have been published in [12]. I would also like to thank Kenneth Stolarsky for pointing out that Part 1 of Corollary 4.1.3 already appeared in [11] and for suggesting more number-theoretic properties to study.

The UIUC Math department has provided me with generous support over the years along with a very stimulating atmosphere. Thus I would like to thank the department along with all the other graduate students that have helped in my mathematical development. In particular, I would like to thank Artan Sheshmani, Pradthana Jaipong, David Lipsky, and Brett Miller for being great friends and for tolerating my obsession with group-invariant CR mappings. Finally, I'd like to thank my parents, Gary and Margaret Grundmeier, for always encouraging my love of mathematics.

Table of Contents

Chapter 1

Introduction

We consider several problems in an emerging field at the intersection of CR-geometry, representation theory, and combinatorics. Let $\mathbb{C}^n$ denote complex Euclidean space and $U(n)$ the unitary group. Let $M \subset \mathbb{C}^n$ and $M' \subset \mathbb{C}^N$ be real hypersurfaces. A fundamental question in CR geometry is to determine whether there are non-constant holomorphic mappings, defined on an open set containing M, taking M to M'. It also makes sense to consider maps holomorphic on only one side of M with some regularity on M. A natural starting place is mappings between spheres. Let S^{2n-1} denote the unit sphere in $\mathbb{C}^n$. When $N > n \geq 2$, there are many non-constant holomorphic mappings f for which $f(S^{2n-1}) \subset S^{2N-1}$. The restriction of f to the sphere is a CR mapping. We naturally ask whether there are such f that respect a group action.

Let Γ be a finite subgroup of the unitary group $U(n)$. We seek CR mappings $f : S^{2n-1} \to S^{2N-1}$ where $f \circ \gamma = f$ for all $\gamma \in \Gamma$. Forstnerič showed, for $n \geq 2$, that a smooth CR mapping from S^{2n-1} to S^{2N-1} must be a rational mapping [10]. The regularity of f is important. Forstnerič showed that if Γ is fixed-point-free, then there is always a continuous, non-constant CR mapping from S^{2n-1}/Γ to S^{2N-1} for some N. Typically such maps extend holomorphically to only one side of the sphere. Thus, other than being fixed-point-free, there are no restrictions on the groups that admit non-constant, continuous CR mappings to spheres. Using Wolf's classification of spherical space forms, Forstnerič found restrictions on the groups that admit rational CR mappings to a sphere. Lichtblau [14] proved the crucial result that for non-constant Γ-invariant rational maps between spheres to exist, Γ must be cyclic. D'Angelo and Lichtblau [2, 7] found the complete list of cyclic $\Gamma \subset U(n)$ for which such a rational map exists. Let I_n denote the n by n identity matrix. They showed that Γ must be equivalent to one of the following groups:

1. $\langle \omega I_n \rangle$ where ω is a primitive m-th root of unity for arbitrary m,

2. $\langle \omega I_j \oplus \omega^2 I_{n-j} \rangle$ where ω is a primitive odd root of unity,

3. $\langle \omega I_j \oplus \omega^2 I_k \oplus \omega^4 I_{n-j-k} \rangle$ where $\omega^7 = 1$.

Thus, we have a complete classification of subgroups of $U(n)$ that admit non-constant, smooth, invariant CR mappings to spheres. Since not all representations of cyclic groups in $U(n)$ admit non-constant, smooth

CR mappings to spheres, we will explore the dependence on the representation.

A natural generalization allows negative eigenvalues in the defining equation of the target. Let $Q(A,B)$ denote the hyperquadric with A positive eigenvalues and B negative eigenvalues in its defining equation; that is

$$Q(A,B) = \left\{ z \in \mathbb{C}^n : \sum_{j=1}^{A} |z_j|^2 - \sum_{j=A+1}^{A+B} |z_j|^2 = 1 \right\}. \tag{1.1}$$

We now ask whether there are non-constant rational CR mappings from S^{2n-1}/Γ to $Q(A,B)$.

In [7], D'Angelo and Lichtblau gave a method of constructing invariant mappings. Following [7], we define the Hermitian polynomial Φ_Γ by

$$\Phi_\Gamma(z,\overline{z}) = 1 - \prod_{\gamma \in \Gamma} (1 - \langle \gamma z, z \rangle). \tag{1.2}$$

In [3], D'Angelo proves the following uniqueness theorem. In Chapter 4, we use this theorem while studying the generating function associated to Φ_Γ for certain cyclic Γ.

Theorem. Φ_Γ *is uniquely determined by the following 4 properties:*

1. $\Phi_\Gamma(0,0) = 0$.
2. $\Phi_\Gamma(\gamma z, \overline{z}) = \Phi_\Gamma(z,\overline{z})$ *for all* $\gamma \in \Gamma$ *and* $z \in \mathbb{C}^n$.
3. $\Phi_\Gamma(z,\overline{z}) = 1$ *when* $||z||^2 = 1$.
4. *The degree of* Φ_Γ *in the* z *variables is at most* $|\Gamma|$.

Expanding the product in (1.2), we write

$$\Phi_\Gamma(z,\overline{z}) = \sum c_{\alpha\beta} z^\alpha \overline{z}^\beta. \tag{1.3}$$

By diagonalizing the underlying Hermitian matrix $(c_{\alpha\beta})$ of coefficients, we get a canonical group-invariant CR mapping from a sphere to a hyperquadric. Diagonalizing $(c_{\alpha\beta})$ allows us to decompose Φ_Γ in the following way

$$\Phi_\Gamma = ||F||^2 - ||G||^2, \tag{1.4}$$

where F and G are holomorphic Γ-invariant polynomials with linearly independent components. Let N^+ denote the number of components of F, and let N^- denote the number of components of G. We obtain the following canonical polynomial mapping:

$$h_\Gamma = F \oplus G : S^{2n-1}/\Gamma \to Q(N^+, N^-) = Q_\Gamma. \tag{1.5}$$

In the rest of this thesis, we study properties of the Hermitian polynomial Φ_Γ. In Chapter 2, we determine Q_Γ for each finite subgroup Γ of $SU(2)$. In Chapter 3, we study the asymptotic behavior of h_Γ for families of cyclic and dihedral subgroups of $U(2)$. In Chapter 4, we study number-theoretic and combinatorial properties of the Φ_Γ associated to mappings of lens spaces $L(p,q)$. In particular, we show that $\Phi_{\Gamma(p,q)}$ satisfies a linear recurrence relation of order $2^q - 1$ and no smaller. Finally, in Chapter 5, we make a connection between the Φ_Γ and invariant theory. We express Φ_Γ as an alternating sum of orbit Chern classes, and we use Molien's theorem to given an upper bound on the total number of eigenvalues in the target hyperquadric associated to Φ_Γ.

1.1 Statements of Results

Let $R : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be a polynomial. We call R Hermitian symmetric if

$$R(z, \overline{w}) = \overline{R(w, \overline{z})}. \tag{1.6}$$

The polynomial

$$r(z, \overline{w}) = \sum c_{\alpha\beta} z^\alpha \overline{w}^\beta \tag{1.7}$$

is Hermitian symmetric if and only if the matrix $(c_{\alpha\beta})$ of coefficients is Hermitian symmetric if and only if $r(z, \overline{z})$ is real-valued (see [2]). We define $N(r)$, $N^+(r)$, and $N^-(r)$ to be the numbers of total eigenvalues, positive eigenvalues, and negative eigenvalues respectively of $(c_{\alpha\beta})$. We define the *signature pair* $S(r)$ of a Hermitian polynomial to be the pair $S(r) = (N^+(r), N^-(r))$. The *signature pair* $S(\Gamma)$ is

$$S(\Gamma) = (N^+(\Phi_\Gamma), N^-(\Phi_\Gamma)). \tag{1.8}$$

Our first result determines the signature pair for finite subgroups of $SU(2)$. Each finite subgroup of $SU(2)$ is isomorphic to one of the following (see [20]):

- Cyclic group of order p: $C_p = \langle a \mid a^p = 1 \rangle$.

- Binary Dihedral group of order $4p$:

$$Q_p = \left\langle a, b \,|a^p = b^2, a^{2p} = 1, b^{-1}ab = a^{-1} \right\rangle. \tag{1.9}$$

- Binary Tetrahedral group of order 24: $T = \left\langle a, b \mid a^3 = b^3 = (ab)^2 \right\rangle$.

- Binary Octahedral group of order 48: $O = \left\langle a, b \mid a^4 = b^3 = (ab)^2 \right\rangle$.

- Binary Icosahedral group of order 120: $I = \left\langle a, b \mid a^5 = b^3 = (ab)^2 \right\rangle$.

In the next theorem, we give the signature pair for each Γ in $SU(2)$, thus determining Q_Γ. We use Mathematica [13] to obtain the signature pair for the binary polyhedral groups. Here is the complete story for all the finite subgroups of $SU(2)$:

Theorem 1.1.1. *Let Γ be a finite subgroup of $SU(2)$.*

1. *If Γ is isomorphic to a cyclic group of order p, then*

$$S(\Gamma) = \left(\left\lfloor \frac{p+2}{4} \right\rfloor + 2, \left\lfloor \frac{p}{4} \right\rfloor \right). \tag{1.10}$$

2. *If Γ is isomorphic to a binary dihedral group of order $4p$, then*

$$S(\Gamma) = \left(\left\lfloor \frac{p}{2} \right\rfloor + p + 2, \left\lfloor \frac{p-1}{2} \right\rfloor + 1 \right). \tag{1.11}$$

3. *If Γ is isomorphic to a binary tetrahedral group of order 24, then*

$$S(\Gamma) = (9, 5). \tag{1.12}$$

4. *If Γ is isomorphic to a binary octahedral group of order 48, then*

$$S(\Gamma) = (17, 9). \tag{1.13}$$

5. *If Γ is isomorphic to a binary icosahedral group of order 120, then*

$$S(\Gamma) = (40, 22). \tag{1.14}$$

We next turn to the finite subgroups of $U(2)$. First we study the cyclic and dihedral subgroups of $U(2)$. For a cyclic subgroup of order p in $U(2)$, several different signature pairs are possible. For dihedral subgroups of $U(2)$, the signature pair depends on only the isomorphism type of the group. For the cyclic group C_p with p elements, we consider the group representations $\pi : C_p \to \Gamma(p,q) < U(2)$ generated by

$$s \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^q \end{pmatrix}, \tag{1.15}$$

where ω is a primitive p-th root of unity and s generates C_p. Up to conjugation, every fixed-point-free finite cyclic subgroup of $U(2)$ is of the form $\Gamma(p,q)$ for some p and q. In this case, it is difficult to compute the signature pair exactly; nonetheless, the next result determines the asymptotic behavior.

Theorem 1.1.2. *Let* $\Gamma(p,q)$ *be as in* (1.15)*, then*

$$\lim_{p\to\infty} \frac{N^+(\Gamma(p,q))}{N(\Gamma(p,q))} = \begin{cases} \frac{3q+1}{4q} & \text{if } q \text{ is odd,} \\ \frac{3q-2}{4(q-1)} & \text{if } q \text{ is even,} \end{cases} \tag{1.16}$$

and hence

$$\lim_{q\to\infty} \lim_{p\to\infty} \frac{N^+(\Gamma(p,q))}{N(\Gamma(p,q))} = \frac{3}{4}. \tag{1.17}$$

The details of the proof appear in Chapter 3, but we give a short description now. First we recall from [4] the weight of a monomial appearing in $\Phi_{\Gamma(p,q)}$. Then we find bounds for the total number of terms and the number of terms of each weight. Using this information, we calculate bounds on the fraction of terms of odd weight and the fraction of terms of even weight. Then we show that asymptotically the numbers of even and odd weight terms are equal. We then interpret a result of Loehr, Warrington, and Wilf [15] in terms of weights. Their result implies that all the odd weight terms are positive and the even weight terms alternate sign. Since $\Gamma(p,q)$ is diagonal, the number of terms is the same as the number of eigenvalues. It follows that the limit as q goes to infinity of the asymptotic positivity ratio is $\frac{3}{4}$.

The third main result calculates the signature pair for dihedral subgroups of $U(2)$.

Theorem 1.1.3. *Let* Δ_p *be a dihedral subgroup of order* $2p$ *in* $U(2)$*, then*

$$S(\Delta_p) = \left(\left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{p}{4} \right\rfloor + 2, \left\lfloor \frac{3(p+1)}{4} \right\rfloor \right), \tag{1.18}$$

and hence

$$\lim_{p\to\infty} L(\Delta_p) = \frac{1}{2}. \tag{1.19}$$

We next study some number-theoretic properties of the Φ_Γ associated to the lens spaces $L(p,q)$. Let $\Gamma(p,q)$ be the cyclic subgroup of $U(2)$ described above. We define $f_{p,q}$ by

$$f_{p,q}(|z|^2, |w|^2) = f_{p,q}(x,y) = 1 - \prod_{j=0}^{p-1} \left(1 - \omega^j x - \omega^{qj} y\right) = \Phi_{\Gamma(p,q)}(z,\overline{z}). \tag{1.20}$$

The polynomials $f_{p,q}$ have many remarkable properties (see [4, 6, 15, 16]). In Chapter 4, we prove the following:

Theorem 1.1.4. *For fixed q, the polynomials $f_{p,q}$ satisfy a linear recurrence relation of order $2^q - 1$ and no smaller. Furthermore, the sequence of polynomials $f_{p,q}$ has a generating function which is rational in $|z|^2$, $|w|^2$, and the indeterminate t.*

Using circulant determinants, Loehr, Warrington, and Wilf [15] gave a combinatorial interpretation of the coefficients of $f_{p,q}$. They left open the problem of finding explicit formulas for the coefficients. In Chapter 4, we find an explicit formula for the coefficients in certain cases.

We next describe how Φ_Γ arises in invariant theory. Theorem 1.1.5 expresses Φ_Γ in terms of orbit Chern classes (as defined in [18]). Let $\pi : G \to U(n)$ be a representation of a finite group G. Let $\mathbb{C}[z_1, \cdots, z_n]$ denote the polynomial algebra in n variables over $\mathbb{C}$. We define a group action on the polynomial algebra by

$$(g\cdot h)(z_1,\cdots,z_n) = h(\pi(g^{-1})(z_1,\cdots,z_n)) \tag{1.21}$$

where $h \in \mathbb{C}[z_1,\cdots,z_n]$ and $g \in G$. The set of fixed points of this action is the set S of group-invariant polynomials in $\mathbb{C}[z_1,\cdots,z_n]$. Thus

$$S = \mathbb{C}[z_1,\cdots,z_n]^G = \left\{h \in \mathbb{C}[z_1,\cdots,z_n] : g\cdot h = h,\ \forall g \in G\right\}. \tag{1.22}$$

Define $G\cdot h$ to be the G-orbit corresponding to $h \in \mathbb{C}[z_1,\cdots,z_n]$. Following [18], define the orbit polynomial of $G\cdot h$ by

$$\phi_{G\cdot h}(X) = \prod_{b\in G\cdot h} (X+b). \tag{1.23}$$

Expanding the product we get

$$\phi_{G\cdot h}(X) = \sum_{a+b=|G|} c_a(G\cdot h) X^b \tag{1.24}$$

where $c_a(G \cdot h) \in \mathbb{C}[z_1, \cdots, z_n]^G$ define the *orbit Chern classes* of the orbit $G \cdot h$.

Theorem 1.1.5. *Let $\pi : G \to U(n)$ be a faithful, unitary representation of the finite group G. Put $\Gamma = \pi(G)$. Then*

$$\Phi_\Gamma(z, 1) = \sum_{j=1}^{p} (-1)^{j-1} c_j(G \cdot (z_1 + \cdots + z_n)). \tag{1.25}$$

Finally we use Molien's theorem to study the total number of eigenvalues of Φ_Γ. Let J^p denote the p-th order Taylor polynomial of f at 0. We give the following upper bound.

Theorem 1.1.6. *If Γ is a finite subgroup of $U(n)$ with order p, then*

$$N(\Gamma) + 1 \leq J^p \left(\frac{1}{p} \sum_{\gamma \in \Gamma} \frac{1}{\det(I_n - t\gamma)} \right) \Bigg|_{t=1}. \tag{1.26}$$

Equality holds in (1.26) for all subgroups of $U(2)$ we consider but not in general. We conclude Chapter 5 by giving an example of a cyclic subgroup of $U(3)$ for which equality fails.

Chapter 2

Signature Pairs of Finite Subgroups of $SU(2)$

In this chapter we study the signature pair $S(\Gamma)$ for various groups Γ. In section 1 we give relevant definitions, introduce the weight of a polynomial, and prove some basic facts about Hermitian polynomials. In section 2 we prove Theorem 1.1.1.

2.1 Definitions and Preliminaries

In this section we recall some basic facts about unitary representations and Hermitian polynomials. Let r be a Hermitian polynomial. In the introduction, we defined $N^+(r)$, $N^-(r)$, $N(r)$, and $S(r)$. We now define the *positivity ratio* by $L(r) = \frac{N^+(r)}{N(r)}$, and $L(\Gamma) = \frac{N^+(\Gamma)}{N(\Gamma)}$.

For families of subgroups Γ_p of $U(n)$, we define the *asymptotic positivity ratio* of Γ_p to be $\lim_{p\to\infty} L(\Gamma_p)$.

Definition 2.1.1. Let C_p be a cyclic group of order p with generator s. Let ω be a p-th primitive root of unity. Define a unitary representation $\pi : C_p \to U(2)$ by

$$\pi(s) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^q \end{pmatrix}. \tag{2.1}$$

Let $\Gamma(p, q) = \pi(C_p)$.

Definition 2.1.2. Group representations $\pi_1 : G \to U(n)$ and $\pi_2 : G \to U(n)$ are called *equivalent* if there exists an element $A \in U(n)$ such that $A\pi_1(g)A^{-1} = \pi_2(g)$ for every $g \in G$.

The following basic property of Hermitian polynomials will be needed in the next section. The signature pair for Hermitian polynomials is unchanged under a change of basis, and therefore equivalent representations have the same signature pair.

Proposition 2.1.3. *If r is a Hermitian polynomial, then $S(r) = S(r \circ U)$ for every U in $U(n)$.*

Proof. Let $r(z,\overline{z})$ be a Hermitian polynomial of degree d; using multi-index notation, we write

$$r(z,\overline{z}) = \sum_{|\alpha|,|\beta|\leq d} c_{\alpha\beta} z^{\alpha}\overline{z}^{\beta}. \tag{2.2}$$

The polynomial $r(z,\overline{z})$ is a Hermitian form on the vector space of polynomials of degree at most d. Composing with U gives

$$r(Uz,\overline{Uz}) = \sum c_{\alpha\beta}(Uz)^{\alpha}(\overline{Uz})^{\beta}. \tag{2.3}$$

Since U is non-singular and the monomials z^{α} form a basis of the vector space of polynomials of degree less than d in z, then $(Uz)^{\alpha}$ also form a basis of the vector space of polynomials of degree at most d. Thus by Sylvester's Law of Inertia, $r(z,\overline{z})$ and $r(Uz,\overline{Uz})$ have the same numbers of eigenvalues of each sign. ☐

Corollary 2.1.4. *If $\pi_1 : G \to U(n)$ and $\pi_2 : G \to U(n)$ are equivalent representations, then $S(\pi_1(G)) = S(\pi_2(G))$.*

Proof. The result follows by a change of coordinates. Let $g \in G$, then there exists $A \in U(n)$ such that $A\pi_1(g)A^{-1} = \pi_2(g)$. Thus, since $(A^{-1})^* = A$,

$$\begin{aligned}
\Phi_{\pi_1(G)}(z,\overline{z}) &= 1 - \prod_{\gamma\in\pi_1(G)}(1 - \langle \gamma z, z\rangle) \\
&= 1 - \prod_{g\in G}(1 - \langle \pi_1(g)z, z\rangle) \\
&= 1 - \prod_{g\in G}\left(1 - \langle A^{-1}\pi_2(g)Az, z\rangle\right) \\
&= 1 - \prod_{g\in G}(1 - \langle \pi_2(g)Az, Az\rangle) \\
&= 1 - \prod_{g\in G}(1 - \langle \pi_2(g)w, w\rangle) \\
&= \Phi_{\pi_2(G)}(w,\overline{w}),
\end{aligned} \tag{2.4}$$

where $w = Az$. By Proposition 2.1.3, the signature pair of a Hermitian polynomial is invariant under a change of coordinates, and the result follows. ☐

2.2 Subgroups of $SU(2)$

For fixed n, we naturally ask what are the finite subgroups of $U(n)$. For $n = 1$, the only finite subgroups are cyclic. For $n > 1$, the question becomes difficult. In this chapter, we restrict to the case $n = 2$. When

$n = 2$, Du Val [9] classified the finite subgroups while studying what are now called Du Val singularities. He found nine families of subgroups. When we restrict to $SU(2)$, the following five types of subgroups arise:

- Cyclic group of order p: $C_p := \langle a \mid a^p = 1 \rangle$.
- Binary Dihedral group of order $4p$: $Q_p := \left\langle a, b \mid a^p = b^2, a^{2p} = 1, b^{-1}ab = a^{-1} \right\rangle$.
- Binary Tetrahedral group of order 24: $T := \left\langle a, b \mid a^3 = b^3 = (ab)^2 \right\rangle$.
- Binary Octahedral group of order 48: $O := \left\langle a, b \mid a^4 = b^3 = (ab)^2 \right\rangle$.
- Binary Icosahedral group of order 120: $I := \left\langle a, b \mid a^5 = b^3 = (ab)^2 \right\rangle$.

In the following sections, we determine the signature pair for each of the finite subgroups of $SU(2)$, thereby proving Theorem 1.1.1.

2.2.1 Cyclic Groups

In this section, we establish part 1 of Theorem 1.1.1. Let Γ be a cyclic subgroup of order p in $SU(2)$. Let A be a generator of Γ in $SU(2)$. By the results of section 2, we can diagonalize A without affecting the signature pair. Thus it suffices to consider A of the form

$$\begin{pmatrix} \omega^a & 0 \\ 0 & \omega^b \end{pmatrix}$$

where ω is a p-th root of unity. Since A is in $SU(2)$, we also know that $a + b = p$. Moreover, for A to have order p, then a, b, and p must be relatively prime. Then a, $p - a$, and p are relatively prime and hence $\omega^{aj} = \omega$ for some j. Thus we can always choose a to be 1 without loss of generality. Hence $b = p - 1$, and then A generates $\Gamma(p, p-1)$. Therefore up to conjugation, $\Gamma(p, p-1)$ is the only cyclic subgroup of order p in $SU(2)$. Thus the only possible signature pair for a cyclic subgroup of order p in $SU(2)$ is given by $\Gamma(p, p-1)$. We recall some useful facts about $\Phi_{\Gamma(p,p-1)}$ from [3]. We will use these properties for computing the asymptotic positivity ratio for other groups.

Theorem 2.2.1. [3] *The following hold for* $\Phi_{\Gamma(p,p-1)}$.

1. *The following exact formula holds:*

$$\Phi_{\Gamma(p,p-1)}(z,\overline{z}) = 1 + |z_1|^{2p} + |z_2|^{2p} - \left(\frac{1+\sqrt{1-4|z_1|^2|z_2|^2}}{2}\right)^p - \left(\frac{1-\sqrt{1-4|z_1|^2|z_2|^2}}{2}\right)^p. \tag{2.5}$$

2. *The coefficients* κ_j *in the following formula are positive integers:*

$$\Phi_{\Gamma(p,p-1)}(z,\overline{z}) = |z_1|^{2p} + |z_2|^{2p} + \sum_{j=1}^{\lfloor \frac{p}{2} \rfloor} (-1)^{j-1}\kappa_j |z_1|^{2j}|z_2|^{2j}. \tag{2.6}$$

3. *These coefficients are given by*

$$\kappa_j(p) = \kappa_j = \frac{p}{p-j}\binom{p-j}{j}. \tag{2.7}$$

4. *Finally the signature pair is*

$$S(\Gamma(p,p-1)) = \left(\left\lfloor \frac{p+2}{4} \right\rfloor + 2, \left\lfloor \frac{p}{4} \right\rfloor\right). \tag{2.8}$$

Formula (2.8) establishes part 1 of Theorem 1.1.1.

Remark 2.2.2. D'Angelo also showed that the coefficients of $f_{p,p-1}$ are the same as the coefficients of $f_{p,2}$ up to sign. More generally the coefficients of $f_{p,q}$ are the same as the coefficients of $f_{p,p-q+1}$ up to sign. We prove this statement in Chapter 4.

Corollary 2.2.3. *The asymptotic positivity ratio for cyclic subgroups of* $SU(2)$ *equals* $\frac{1}{2}$. *Thus:*

$$\lim_{p\to\infty} L(\Gamma(p,p-1)) = \frac{1}{2}. \tag{2.9}$$

2.2.2 Binary Dihedral Groups

In this section, we prove part 2 of Theorem 1.1.1. Consider the binary dihedral groups

$$Q_p := \left\langle a, b \,| a^p = b^2, a^{2p} = 1, b^{-1}ab = a^{-1} \right\rangle. \tag{2.10}$$

The group Q_p has order $4p$. Let $\eta : Q_p \to SU(2)$ be the faithful representation generated by

$$\eta(a) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \tag{2.11}$$

$$\eta(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.12}$$

where ω is a $2p$-th primitive root of unity. Set $\Lambda_p = \eta(Q_p)$.

Before stating the main results of this section we illustrate the techniques with an example. We compute the number of positive and negative eigenvalues of Φ_{Λ_2}. Expanding the product in the definition we get

$$\begin{aligned}\Phi_{\Lambda_2}(z,\overline{z}) = {} & z_1^4\overline{z_1}^4 + z_2^4\overline{z_1}^4 - z_1^4 z_2^4\overline{z_1}^8 + 4z_1^5 z_2\overline{z_1}^5\overline{z_2} - 4z_1 z_2^5\overline{z_1}^5\overline{z_2} + 12z_1^2 z_2^2\overline{z_1}^2\overline{z_2}^2 \\ & + 2z_1^6 z_2^2\overline{z_1}^6\overline{z_2}^2 + 2z_1^2 z_2^6\overline{z_1}^6\overline{z_2}^2 + z_1^4\overline{z_2}^4 + z_2^4\overline{z_2}^4 - z_1^8\overline{z_1}^4\overline{z_2}^4 - 4z_1^4 z_2^4\overline{z_1}^4\overline{z_2}^4 \\ & - z_2^8\overline{z_1}^4\overline{z_2}^4 - 4z_1^5 z_2\overline{z_1}\,\overline{z_2}^5 + 4z_1 z_2^5\overline{z_1}\,\overline{z_2}^5 + 2z_1^6 z_2^2\overline{z_1}^2\overline{z_2}^6 + 2z_1^2 z_2^6\overline{z_1}^2\overline{z_2}^6 - z_1^4 z_2^4\overline{z_2}^8.\end{aligned} \tag{2.13}$$

Notice that in contrast to the cyclic case, we get off-diagonal terms, so it is not enough to simply count the number of terms to get the number of eigenvalues. Rewriting in terms of polynomials invariant under the Q_2-action, we get

$$\begin{aligned}\Phi_{\Lambda_2} = {} & (z_1^4 + z_2^4)(\overline{z_1}^4 + \overline{z_2}^4) - 4z_1^4 z_2^4\overline{z_1}^4\overline{z_2}^4 + 12z_1^2 z_2^2\overline{z_1}^2\overline{z_2}^2 + 4z_1 z_2(z_1^4 - z_2^4)\overline{z_1}\,\overline{z_2}(\overline{z_1}^4 - \overline{z_2}^4) \\ & + 2z_1^2 z_2^2(z_1^4 + z_2^4)\overline{z_1}^2\overline{z_2}^2(\overline{z_1}^4 + \overline{z_2}^4) - z_1^4 z_2^4(\overline{z_1}^8 + \overline{z_2}^8) - \overline{z_1}^4\overline{z_2}^4(z_1^8 + z_2^8).\end{aligned} \tag{2.14}$$

Equivalently we get

$$\Phi_{\Lambda_2} = \begin{pmatrix} \overline{z_1}^4 + \overline{z_2}^4 \\ \overline{z_1}\,\overline{z_2}(\overline{z_1}^4 - \overline{z_2}^4) \\ \overline{z_1}^2\overline{z_2}^2(\overline{z_1}^4 + \overline{z_2}^4) \\ \overline{z_1}^2\overline{z_2}^2 \\ \overline{z_1}^4\overline{z_2}^4 \\ \overline{z_1}^8 + \overline{z_2}^8 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} z_1^4 + z_2^4 \\ z_1 z_2(z_1^4 - z_2^4) \\ z_1^2 z_2^2(z_1^4 + z_2^4) \\ z_1^2 z_2^2 \\ z_1^4 z_2^4 \\ z_1^8 + z_2^8 \end{pmatrix}. \tag{2.15}$$

Hence the eigenvalues of Φ_{Λ_2} are 1, 4, 2, 12, $-2+\sqrt{5}$, $-2-\sqrt{5}$. Therefore $S(\Lambda_2) = (5,1)$.

For clarity, we explicitly write Φ_{Λ_2} as a difference of squared norms. Let

$$A(z) = \begin{pmatrix} z_1^4 + z_2^4 \\ 2z_1 z_2 (z_1^4 - z_2^4) \\ \sqrt{2} z_1^2 z_2^2 (z_1^4 + z_2^4) \\ \sqrt{12} z_1^2 z_2^2 \\ (\sqrt{-2+\sqrt{5}})(z_1^8 + (2-\sqrt{5}) z_1^4 z_2^4 + z_2^8) \end{pmatrix}, \tag{2.16}$$

and

$$B(z) = \left((\sqrt{2+\sqrt{5}})(z_1^8 + (2+\sqrt{5}) z_1^4 z_2^4 + z_2^8) \right). \tag{2.17}$$

Then $\Phi_{\Lambda_2} = ||A(z)||^2 - ||B(z)||^2$.

We proceed along these lines for general p. The next proposition relates Φ_{Λ_p} to the cyclic case.

Proposition 2.2.4. *The invariant polynomial corresponding to the representation η satisfies:*

$$\begin{aligned} \Phi_{\Lambda_p} &= f_{2p,2p-1}(|z_1|^2, |z_2|^2) + f_{2p,2p-1}(z_2\overline{z_1}, -z_1\overline{z_2}) \\ &\quad - f_{2p,2p-1}(|z_1|^2, |z_2|^2) f_{2p,2p-1}(z_2\overline{z_1}, -z_1\overline{z_2}). \end{aligned} \tag{2.18}$$

Proof. As we alluded to above, the key idea is to notice that

$$\Lambda_p = \left\{ \begin{pmatrix} \omega^j & 0 \\ 0 & \omega^{-j} \end{pmatrix}, \begin{pmatrix} 0 & \omega^j \\ -\omega^{-j} & 0 \end{pmatrix} : j = 0, \cdots, 2p-1 \right\}. \tag{2.19}$$

We consider the factors separately:

$$\begin{aligned} \Phi_{\Lambda_p}(z, \overline{z}) &= 1 - \prod_{\gamma \in \Lambda_p} (1 - \langle \gamma z, z \rangle) \\ &= 1 - \left(\prod_{j=0}^{2p-1} \left(1 - \left\langle \begin{pmatrix} \omega^j & 0 \\ 0 & \omega^{-j} \end{pmatrix} z, z \right\rangle \right) \right) \left(\prod_{j=0}^{2p-1} \left(1 - \left\langle \begin{pmatrix} 0 & \omega^j \\ -\omega^{-j} & 0 \end{pmatrix} z, z \right\rangle \right) \right) \\ &= 1 - \left(\prod_{j=0}^{2p-1} \left(1 - \omega^j z_1 \overline{z_1} - \omega^{-j} z_2 \overline{z_2} \right) \right) \left(\prod_{j=0}^{2p-1} \left(1 - \omega^j z_2 \overline{z_1} + \omega^{-j} z_1 \overline{z_2} \right) \right). \end{aligned} \tag{2.20}$$

Notice that each product corresponds to the polynomial $f_{2p,2p-1}$ evaluated at different points:

$$\begin{aligned}\Phi_{\Lambda_p}(z,\overline{z}) &= 1-\left(1-f_{2p,2p-1}(|z_1|^2,|z_2|^2)\right)\left(1-f_{2p,2p-1}(z_2\overline{z_1},-z_1\overline{z_2})\right)\\ &= f_{2p,2p-1}(|z_1|^2,|z_2|^2)+f_{2p,2p-1}(z_2\overline{z_1},-z_1\overline{z_2})\\ &\qquad -f_{2p,2p-1}(|z_1|^2,|z_2|^2)f_{2p,2p-1}(z_2\overline{z_1},-z_1\overline{z_2}).\end{aligned}\tag{2.21}$$

This formula completes the proof. □

Now we proceed by using the previous theorem to express the polynomial Φ_{Λ_p} in terms of the $f_{2p,2p-1}$ polynomials. The goal then is to express Φ_{Λ_p} in terms of the following linearly independent Q_p-invariant polynomials:

$$z_1^{2p}+z_2^{2p},\ z_1^j z_2^j(z_1^{2p}+(-1)^j z_2^{2p}),\ (z_1z_2)^{2j}, z_1^{4p}+z_2^{4p}\tag{2.22}$$

for $j=1,\cdots,p$.

For the reader's convenience, we again recall

$$f_{2p,2p-1}(x,y)=x^{2p}+y^{2p}+\sum_{j=1}^{p}(-1)^{j-1}\kappa_j(2p)(xy)^j.\tag{2.23}$$

Then by Proposition 2.2.4

$$\begin{aligned}\Phi_{\Lambda_p}(z,\overline{z}) &= (z_1\overline{z_1})^{2p}+(z_2\overline{z_2})^{2p}+\sum_{j=1}^{p}(-1)^{j-1}\kappa_j(2p)(z_1z_2\overline{z_1z_2})^j+(z_2\overline{z_1})^{2p}+(z_1\overline{z_2})^{2p}\\ &\quad+\sum_{j=1}^{p}(-1)\kappa_j(2p)(z_1z_2\overline{z_1z_2})^j-z_1^{2p}z_2^{2p}\overline{z_1}^{4p}-z_1^{4p}\overline{z_1}^{2p}\overline{z_2}^{2p}-z_2^{4p}\overline{z_1}^{2p}\overline{z_2}^{2p}\\ &\quad-z_1^{2p}z_2^{2p}\overline{z_2}^{4p}+\sum_{j=1}^{p}\kappa_j(2p)z_1^{2p+j}z_2^j\overline{z_1}^{2p+j}\overline{z_2}^j+\sum_{j=1}^{p}\kappa_j(2p)z_1^jz_2^{2p+j}\overline{z_1}^j\overline{z_2}^{2p+j}\\ &\quad+\sum_{j=1}^{p}(-1)^j\kappa_j(2p)z_1^jz_2^{2p+j}\overline{z_1}^{2p+j}\overline{z_2}^j+\sum_{j=1}^{p}(-1)^j\kappa_j(2p)z_1^{2p+j}z_2^j\overline{z_1}^j\overline{z_2}^{2p+j}\\ &\quad-\left(\sum_{j=1}^{p}(-1)^{j-1}\kappa_j(2p)(z_1z_2\overline{z_1z_2})^j\right)\left(\sum_{j=1}^{p}(-1)\kappa_j(2p)(z_1z_2\overline{z_1z_2})^j\right).\end{aligned}\tag{2.24}$$

Notice that all the odd power terms drop from the product, and we get the following simplification.

$$\left(\sum_{j=1}^{p}(-1)^{j-1}\kappa_j(2p)(z_1z_2\overline{z_1}\overline{z_2})^j\right)\left(\sum_{j=1}^{p}(-1)\kappa_j(2p)(z_1z_2\overline{z_1}\overline{z_2})^j\right) =$$
$$\sum_{j=1}^{p}\left(2\sum_{k=j+1}^{\min(2j-1,p)}(-1)^{k-1}\kappa_k(2p)\kappa_{2j-k}(2p)+(-1)^{j-1}\kappa_j(2p)^2\right)(z_1z_2\overline{z_1}\overline{z_2})^{2j}. \tag{2.25}$$

We write the previous expression in terms of the invariant polynomials given above:

$$\begin{aligned}\Phi_{\Lambda_p}(z,\overline{z}) \quad = \quad & (z_1^{2p}+z_2^{2p})(\overline{z_1}^{2p}+\overline{z_2}^{2p})+\sum_{j=1}^{p}\kappa_j(2p)z_1^jz_2^i(z_1^{2p}+(-1)^jz_2^{2p})\overline{z_1}^j\overline{z_2}^j(\overline{z_1}^{2p}+(-1)^j\overline{z_2}^{2p})\\ & +\sum_{j=1}^{\lfloor\frac{p}{2}\rfloor}\left(2\sum_{k=j+1}^{2j-1}(-1)^{k-1}\kappa_k(2p)\kappa_{2j-k}(2p)+(-1)^{j-1}\kappa_j(2p)^2-2\kappa_{2j}(2p)\right)(z_1z_2\overline{z_1}\overline{z_2})^{2j}\\ & +\sum_{j=\lfloor\frac{p}{2}\rfloor+1}^{p}\left(2\sum_{k=j+1}^{p}(-1)^{k-1}\kappa_k(2p)\kappa_{2j-k}(2p)+(-1)^{j-1}\kappa_k(2p)^2\right)(z_1z_2\overline{z_1}\overline{z_2})^{2j}\\ & -z_1^{2p}z_2^{2p}(\overline{z_1}^{4p}+\overline{z_2}^{4p})-\overline{z_1}^{2p}\overline{z_2}^{2p}(z_1^{4p}+z_2^{4p}).\end{aligned} \tag{2.26}$$

It remains to determine the signs of the coefficients in the above expression. First we take care of the obvious cases. The coefficient $\kappa_j(2p)$ of $|z_1^jz_2^j(z_1^{2p}+(-1)^jz_2^{2p})|^2$ is positive. The coefficient of $|z_1^{2p}+z_2^{2p}|^2$ is also positive. We now consider the coefficient of $|z_1z_2|^{4j}$.

Definition 2.2.5. Define d_k to be the coefficient of $(z_1\overline{z_1}z_2\overline{z_2})^{2k}$ in $\Phi_{\Lambda_p}(z,\overline{z})$. Further we define the polynomial D_p by

$$D_p(t)=\sum_{j=1}^{p}d_kt^{2k}. \tag{2.27}$$

We give an exact formula for $D_p(t)$ in the next proposition.

Proposition 2.2.6. *The polynomial $D_p(t)$ is given by*

$$\begin{aligned}D_p(t)=1 \quad & - \quad \frac{1}{4^p}\big((1+a+b+ab)^{2p}+(1-a+b-ab)^{2p}\\ & + \quad (1+a-b-ab)^{2p}+(1-a-b+ab)^{2p}\big)\end{aligned} \tag{2.28}$$

where $a=\sqrt{1-4t}$ and $b=\sqrt{1+4t}$.

Proof. By Proposition 2.2.4 and Theorem 2.2.1,

$$\begin{aligned}
\Phi_{\Lambda_p} &= 1+|z_1|^{4p}+|z_2|^{4p}-\left(\frac{1+\sqrt{1-4|z_1z_2|^2}}{2}\right)^{2p}-\left(\frac{1-\sqrt{1-4|z_1z_2|^2}}{2}\right)^{2p} \\
&+ 1+(z_2\overline{z_1})^{2p}+(z_1\overline{z_2})^{2p}-\left(\frac{1+\sqrt{1+4|z_1z_2|^2}}{2}\right)^{2p}-\left(\frac{1-\sqrt{1+4|z_1z_2|^2}}{2}\right)^{2p} \\
&- \left(1+|z_1|^{4p}+|z_2|^{4p}-\left(\frac{1+\sqrt{1-4|z_1z_2|^2}}{2}\right)^{2p}-\left(\frac{1-\sqrt{1-4|z_1z_2|^2}}{2}\right)^{2p}\right) \\
&\times \left(1+(z_2\overline{z_1})^{2p}+(z_1\overline{z_2})^{2p}-\left(\frac{1+\sqrt{1+4|z_1z_2|^2}}{2}\right)^{2p}-\left(\frac{1-\sqrt{1+4|z_1z_2|^2}}{2}\right)^{2p}\right). \quad (2.29)
\end{aligned}$$

Next we let $t=|z_1z_2|^2$. Take all the terms involving t in the previous expression to get the following:

$$\begin{aligned}
D_p(t) &= 1-\left(\frac{1+\sqrt{1-4t}}{2}\right)^{2p}-\left(\frac{1-\sqrt{1-4t}}{2}\right)^{2p} \\
&+ 1-\left(\frac{1+\sqrt{1+4t}}{2}\right)^{2p}-\left(\frac{1-\sqrt{1+4t}}{2}\right)^{2p} \\
&- \left(1-\left(\frac{1+\sqrt{1-4t}}{2}\right)^{2p}-\left(\frac{1-\sqrt{1-4t}}{2}\right)^{2p}\right) \\
&\times \left(1-\left(\frac{1+\sqrt{1+4t}}{2}\right)^{2p}-\left(\frac{1-\sqrt{1+4t}}{2}\right)^{2p}\right). \quad (2.30)
\end{aligned}$$

Multiply this expression out to get the desired result:

$$\begin{aligned}
D_p(t) &= 1-\frac{1}{4^p}\Bigg(\left(1+\sqrt{1+4t}\right)^{2p}\left(1+\sqrt{1-4t}\right)^{2p}+\left(1+\sqrt{1-4t}\right)^{2p}\left(1-\sqrt{1+4t}\right)^{2p} \\
&+ \left(1+\sqrt{1+4t}\right)^{2p}\left(1-\sqrt{1-4t}\right)^{2p}+\left(1-\sqrt{1-4t}\right)^{2p}\left(1-\sqrt{1+4t}\right)^{2p}\Bigg). \quad (2.31)
\end{aligned}$$

□

Lemma 2.2.7. *The following identity holds:*

$$\begin{aligned}
&(1+a+b+ab)^{2p}+(1-a+b-ab)^{2p}+(1+a-b-ab)^{2p}+(1-a-b+ab)^{2p} \\
&= 4\sum_{j=0}^{p}\sum_{k=0}^{p}\binom{2p}{2j}\binom{2p}{2k}a^{2j}b^{2k} \quad (2.32)
\end{aligned}$$

Proof. Factoring and using the binomial theorem gives

$$\begin{aligned}
&(1+a+b+ab)^{2p}+(1-a+b-ab)^{2p}+(1+a-b-ab)^{2p}+(1-a-b+ab)^{2p}\\
&=((1+a)(1+b))^{2p}+((1-a)(1+b))^{2p}+((1+a)(1-b))^{2p}+((1-a)(1-b))^{2p}\\
&=\left(\sum_{j=0}^{2p}\binom{2p}{j}a^j\right)\left(\sum_{k=0}^{2p}\binom{2p}{k}b^k\right)+\left(\sum_{j=0}^{2p}\binom{2p}{j}(-1)^j a^j\right)\left(\sum_{k=0}^{2p}\binom{2p}{k}b^k\right)\\
&+\left(\sum_{j=0}^{2p}\binom{2p}{j}a^j\right)\left(\sum_{k=0}^{2p}\binom{2p}{k}(-1)^k b^k\right)+\left(\sum_{j=0}^{2p}\binom{2p}{j}(-1)^j a^j\right)\left(\sum_{k=0}^{2p}\binom{2p}{k}(-1)^k b^k\right). \qquad (2.33)
\end{aligned}$$

After multiplying out the right hand side and collecting terms we get

$$\sum_{j=0}^{2p}\sum_{k=0}^{2p}\binom{2p}{j}\binom{2p}{k}a^j b^k\left(1+(-1)^j+(-1)^k+(-1)^{j+k}\right). \qquad (2.34)$$

Since

$$\left(1+(-1)^j+(-1)^k+(-1)^{j+k}\right)=\begin{cases}4 & \text{if } j \text{ and } k \text{ are both even,}\\ 0 & \text{otherwise,}\end{cases} \qquad (2.35)$$

the identity follows after reindexing. □

Let $a=\sqrt{z}$ and $b=\sqrt{\overline{z}}$ in Lemma 2.2.7; then

$$4\sum_{j=0}^{p}\sum_{k=0}^{p}\binom{2p}{2j}\binom{2p}{2k}z^j\overline{z}^k=\left|2\sum_{k=0}^{p}\binom{2p}{2k}z^k\right|^2. \qquad (2.36)$$

Lemma 2.2.8. *Given a polynomial $p(x+iy)$ with all negative real roots, then $|p(x+iy)|^2$ has positive coefficients.*

Proof. Let $a_0, \cdots, a_d$ be the absolute values of the roots of p. Then expanding and simplifying p, we get

$$\begin{aligned}
|p(x+iy)|^2 &= \left|\prod_{j=0}^{d}(x+iy+a_j)\right|^2\\
&= \prod_{j=0}^{d}(x+iy+a_j)(x-iy+a_j)=\prod_{j=0}^{d}\left(x^2+2xa_j+y^2+a_j^2\right). \qquad (2.37)
\end{aligned}$$

In the last expression only positive coefficients occur, and expanding the product gives the desired result. □

D'Angelo provided me the statement and proof of the following lemma.

Lemma 2.2.9. *The following identity holds:*

$$P(z) = 2\sum_{k=0}^{p}\binom{2p}{2k}z^k = \prod_{j=0}^{p-1}\left(z + \tan^2\left(\frac{(2j+1)\pi}{4p}\right)\right), \tag{2.38}$$

and hence all the roots of P are negative. Therefore the coefficients of P are positive.

Proof. Taking proper care of the choice of square root, we can rewrite P in the following way:

$$P(z) = 2\sum_{k=0}^{p}\binom{2p}{2k}z^k = \left(1-\sqrt{z}\right)^{2p} + \left(1+\sqrt{z}\right)^{2p}. \tag{2.39}$$

Setting the right hand side equal to zero yields

$$\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{2p} = -1. \tag{2.40}$$

Taking $2p$-th roots gives

$$\frac{1-\sqrt{z}}{1+\sqrt{z}} = e^{\frac{(2n+1)\pi i}{2p}} \tag{2.41}$$

for $n = 0, \cdots, 2p-1$.

We solve for $\sqrt{z}$:

$$\sqrt{z} = \frac{1-e^{\frac{(2n+1)\pi i}{2p}}}{1+e^{\frac{(2n+1)\pi i}{2p}}} = \frac{e^{-\frac{(2n+1)\pi i}{4p}} - e^{\frac{(2n+1)\pi i}{4p}}}{e^{-\frac{(2n+1)\pi i}{4p}} + e^{\frac{(2n+1)\pi i}{4p}}} = i\tan\left(\frac{(2n+1)\pi}{4p}\right). \tag{2.42}$$

The roots of P are therefore $-\tan^2\left(\frac{(2j+1)\pi}{4p}\right)$, and hence the identity follows. □

Finally we combine the previous lemmas to determine the sign of d_k (defined in (2.27)).

Proposition 2.2.10. *For $1 \leq k \leq p$,*

1. *$d_k > 0$ for k odd.*

2. *$d_k < 0$ for k even.*

Proof. The following relationship between $D_p(t)$ and $P(z)$ holds:

$$D(it) = 1 - \frac{1}{4^p}P(1+4it). \tag{2.43}$$

By the previous lemmas $P(z)$ has all positive coefficients; thus $D_p(it)$ must have all negative coefficients. Since $D(t)$ is a polynomial with only even powers, the transformation $t \mapsto it$ changes the sign of d_k for odd

k and does not change the sign of d_k when k is even. Therefore d_k must be positive for k odd and negative for k even. $\square$

We rephrase the results in terms of matrices:

$$\Phi_{\Lambda_p}(z,\overline{z}) = d^* M_p d \tag{2.44}$$

where

$$d = \begin{pmatrix} z_1^{2p} + z_2^{2p} \\ z_1 z_2 (z_1^{2p} + (-1) z_2^{2p}) \\ \vdots \\ z_1^p z_2^p (z_1^{2p} + (-1)^p z_2^{2p}) \\ z_1^2 z_2^2 \\ \vdots \\ z_1^{2p} z_2^{2p} \\ z_1^{4p} + z_2^{4p} \end{pmatrix}, \tag{2.45}$$

and

$$M_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & E_{p,1} & 0 & 0 \\ 0 & 0 & E_{p,2} & 0 \\ 0 & 0 & 0 & E_{p,3} \end{pmatrix}. \tag{2.46}$$

Here $E_{p,1}$ is the p by p matrix with $\kappa_j(2p)$ on the diagonal. Also $E_{p,2}$ is the square matrix of size $p-1$ with diagonal entries

$$(E_{p,2})_{jj} = 2\sum_{k=j+1}^{2j-1} (-1)^{k-1}\kappa_k(2p)\kappa_{2j-k}(2p) + (-1)^{j-1}\kappa_j(2p)^2 - 2\kappa_{2j}(2p) \tag{2.47}$$

for $1 \leq j \leq \left\lfloor \frac{p}{2} \right\rfloor$, and

$$(E_{p,2})_{jj} = 2\sum_{k=j+1}^{p} (-1)^{k-1}\kappa_k(2p)\kappa_{2j-k}(2p) + (-1)^{j-1}\kappa_j(2p)^2 \tag{2.48}$$

for $\lfloor \frac{p}{2} \rfloor < j \leq p-1$. Finally, the 2 by 2 matrix $E_{p,3}$ is given by,

$$E_{p,3} = \begin{pmatrix} (-1)^{p-1}\kappa_p(2p)^2 & -1 \\ -1 & 0 \end{pmatrix}. \tag{2.49}$$

Now we are left with the task of computing the signature pair of M_p. We proceed by counting the number of eigenvalues of each sign in the submatrices.

Proposition 2.2.11. *For $1 \leq j \leq p-1$,*

1. *$(E_{p,2})_{jj} > 0$ if j is odd.*

2. *$(E_{p,2})_{jj} < 0$ if j is even.*

Proof. The conclusion follows from Proposition 2.2.10. □

The diagonal matrices $E_{p,1}$ and $E_{p,2}$ have non-zero diagonal entries. The submatrix $E_{p,1}$ has p eigenvalues, all of which are positive. Moreover, by the proposition, the diagonal entries in the matrices $E_{p,2}$ alternate sign. Also the matrix $E_{p,3}$ has one eigenvalue of each sign. Combining these results for the submatrices of M_p, we obtain part 2 of Theorem 1.1.1.

Theorem. *The signature pair of the binary dihedral group with $4p$ elements is given by*

$$S(\Lambda_p) = \left(2 + p + \left\lfloor \frac{p}{2} \right\rfloor, 1 + \left\lfloor \frac{p-1}{2} \right\rfloor\right). \tag{2.50}$$

Taking the limit as p goes to infinity yields the following result.

Corollary 2.2.12. *The asymptotic positivity ratio for Λ_p is $\frac{3}{4}$.*

2.2.3 Binary Tetrahedral Group

In this section, we prove part 3 of Theorem 1.1.1. This result is obtained by using Mathematica. See appendix A for the code. The binary tetrahedral group is given by

$$T := \langle a, b \mid a^3 = b^3 = (ab)^2 \rangle \tag{2.51}$$

and has order 24. We represent T in $SU(2)$ using the Springer description [20]. Let $\epsilon = e^{\frac{\pi i}{4}}$. Define

$$\begin{aligned} r &= \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}; \\ s &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \\ t &= \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon^{-1} & \epsilon^{-1} \\ -\epsilon & \epsilon \end{pmatrix}. \end{aligned} \tag{2.52}$$

Define the faithful unitary representation $\mu : T \to SU(2)$ by

$$\mu(a) = st^{-1} \text{ and } \mu(b) = t. \tag{2.53}$$

Let $\Gamma = \mu(T)$. For all $\gamma \in \Gamma$ we can represent γ in the following way:

$$\gamma = r^{2j} s^k t^l \tag{2.54}$$

for some $0 \le j < 3$, $0 \le k < 2$, and $0 \le l < 3$. We remark that all faithful representations of T in $SU(2)$ are equivalent to the representation given by μ.

We express Φ_Γ in terms in Γ-invariant polynomials. The following 14 linearly independent Γ-invariant polynomials appear:

$$\begin{aligned}
C_1 &= z_1^{16} + 28z_1^{12}z_2^4 + 198z_1^8z_2^8 + 28z_1^4z_2^{12} + z_2^{16} \\
C_2 &= z_1^{20} - 19z_1^{16}z_2^4 - 494z_1^{12}z_2^8 - 494z_1^8z_2^{12} - 19z_1^4z_2^{16} + z_2^{20} \\
C_3 &= z_1^{18}z_2^2 + 12z_1^{14}z_2^6 - 26z_1^{10}z_2^{10} + 12z_1^6z_2^{14} + z_1^2z_2^{18} \\
C_4 &= -z_1^{21}z_2 - 27z_1^{17}z_2^5 - 170z_1^{13}z_2^9 + 170z_1^9z_2^{13} + 27z_1^5z_2^{17} + z_1z_2^{21} \\
C_5 &= -z_1^{15}z_2^3 + 3z_1^{11}z_2^7 - 3z_1^7z_2^{11} + z_1^3z_2^{15} \\
C_6 &= z_1^{12} - 33z_1^8z_2^4 - 33z_1^4z_2^8 + z_2^{12} \\
C_7 &= -z_1^{13}z_2 - 13z_1^9z_2^5 + 13z_1^5z_2^9 + z_1z_2^{13} \\
C_8 &= -z_1^{17}z_2 + 34z_1^{13}z_2^5 - 34z_1^5z_2^{13} + z_1z_2^{17} \\
C_9 &= z_1^8 + 14z_1^4z_2^4 + z_2^8 \\
C_{10} &= z_1^{24} + \left(-\tfrac{4692}{35} + \tfrac{1}{35}\left(2382 + \sqrt{119948010}\right)\right) z_1^{20}z_2^4 + \left(\tfrac{45333}{35} + \tfrac{4}{35}\left(-2382 - \sqrt{119948010}\right)\right) z_1^{16}z_2^8 \\
&+ \left(\tfrac{62008}{35} - \tfrac{6}{35}\left(-2382 - \sqrt{119948010}\right)\right) z_1^{12}z_2^{12} + \left(\tfrac{45333}{35} + \tfrac{4}{35}\left(-2382 - \sqrt{119948010}\right)\right) z_1^8z_2^{16} \\
&+ \left(-\tfrac{4692}{35} + \tfrac{1}{35}\left(2382 + \sqrt{119948010}\right)\right) z_1^4z_2^{20} + z_2^{24}
\end{aligned}$$

$$
\begin{aligned}
C_{11} &= z_1^{10}z_2^2 - 2z_1^6z_2^6 + z_1^2z_2^{10} \\
C_{12} &= z_1^{24} + \left(-\tfrac{4692}{35} + \tfrac{1}{35}\left(2382 - \sqrt{119948010}\right)\right) z_1^{20}z_2^4 + \left(\tfrac{45333}{35} + \tfrac{4}{35}\left(-2382 + \sqrt{119948010}\right)\right) z_1^{16}z_2^8 \\
&+ \left(\tfrac{62008}{35} - \tfrac{6}{35}\left(-2382 + \sqrt{119948010}\right)\right) z_1^{12}z_2^{12} + \left(\tfrac{45333}{35} + \tfrac{4}{35}\left(-2382 + \sqrt{119948010}\right)\right) z_1^8z_2^{16} \\
&+ \left(-\tfrac{4692}{35} + \tfrac{1}{35}\left(2382 - \sqrt{119948010}\right)\right) z_1^4z_2^{20} + z_2^{24} \\
C_{13} &= z_1^{22}z_2^2 - 35z_1^{18}z_2^6 + 34z_1^{14}z_2^{10} + 34z_1^{10}z_2^{14} - 35z_1^6z_2^{18} + z_1^2z_2^{22} \\
C_{14} &= -z_1^5z_2 + z_1z_2^5.
\end{aligned}
$$

Define

$$
A(z) = \begin{pmatrix}
\sqrt{\frac{305805}{128}}C_1 \\
\sqrt{\frac{122199}{64}}C_2 \\
\sqrt{\frac{14815}{16}}C_4 \\
\sqrt{740}C_5 \\
\sqrt{\frac{2725}{4}}C_6 \\
\sqrt{\frac{495}{4}}C_9 \\
\sqrt{\frac{1}{128}(-2382 + \sqrt{119948010})C_{12}} \\
\sqrt{\frac{1191}{32}}C_{13} \\
\sqrt{24}C_{14}
\end{pmatrix} \tag{2.55}
$$

and

$$
B(z) = \begin{pmatrix}
\sqrt{\frac{48783}{32}}C_3 \\
\sqrt{680}C_7 \\
\sqrt{\frac{1157}{2}}C_8 \\
\sqrt{\frac{1}{128}(2382 + \sqrt{119948010})C_{10}} \\
\sqrt{\frac{171}{2}}C_{11}
\end{pmatrix}. \tag{2.56}
$$

Using Mathematica [13] one can verify that $\Phi_\Gamma = ||A(z)||^2 - ||B(z)||^2$. Therefore, $S(\Gamma) = (9, 5)$.

Remark 2.2.13. If $\Gamma < SU(2)$, and Γ is isomorphic to T, then $S(\Gamma) = (9, 5)$. Thus we have established part 3 of Theorem 1.1.1.

2.2.4 Binary Octahedral Group

In this section, we prove part 4 of Theorem 1.1.1. This result also uses the source code in appendix A. The binary octahedral group is given by

$$
O := \left\langle a, b \mid a^4 = b^3 = (ab)^2 \right\rangle \tag{2.57}
$$

and has order 48. We again represent O in $SU(2)$ using the Springer description [20]. Recall the generators of the binary tetrahedral group r, s, and t given above in (2.52). Let $\tau : O \to SU(2)$ be a faithful unitary representation generated by

$$\tau(a) = rt \text{ and } \tau(b) = t. \tag{2.58}$$

Notice

$$(rt)^4 = t^3 = (rt^2)^2 = -1. \tag{2.59}$$

Let $\Gamma = \tau(O)$. For all $\gamma \in \Gamma$ we can represent γ in the following way:

$$\gamma = r^j s^k t^l \tag{2.60}$$

for some $0 \leq j < 8$, $0 \leq k < 2$, and $0 \leq l < 3$. We remark that all faithful representations of O in $SU(2)$ are equivalent to the representation given by τ.

The Γ-invariant polynomial Φ_Γ has 1143 terms. Using Mathematica we decompose

$$\Phi_\Gamma = d^* M d \tag{2.61}$$

where M is the Hermitian coefficient matrix and d is the vector of 135 monomials that appear in Φ_Γ. Again using Mathematica we find that M has rank 26 with 17 positive eigenvalues and 9 negative eigenvalues.

Remark 2.2.14. If $\Gamma < SU(2)$, and Γ is isomorphic to O, then $S(\Gamma) = (17, 9)$. Thus we have established part 4 of Theorem 1.1.1.

2.2.5 Binary Icosahedral Group

In this section, we use Mathematica to prove part 5 of Theorem 1.1.1. Again see appendix A for the Mathematica code. The binary icosahedral group is given by

$$I := \left\langle a, b \mid a^5 = b^3 = (ab)^2 \right\rangle \tag{2.62}$$

and has order 120. We again represent I in $SU(2)$ using the Springer description [20]. Let $\epsilon = e^{\frac{2\pi i}{5}}$. Define

$$r = -\begin{pmatrix} \epsilon^3 & 0 \\ 0 & \epsilon^2 \end{pmatrix};$$
$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \tag{2.63}$$
$$t = \frac{1}{\epsilon^2 - \epsilon^{-2}} \begin{pmatrix} \epsilon + \epsilon^{-1} & 1 \\ 1 & -\epsilon - \epsilon^{-1} \end{pmatrix}.$$

Then we define a representation of I in $SU(2)$ by $a = r$ and $b = r^4 ts$. Notice that

$$(r)^5 = (r^4 ts)^3 = (r^5 ts)^2 = -1. \tag{2.64}$$

In [20], Springer describes the 120 elements in the binary icosahedral group as follows:

$$\Gamma = \left\{ r^h, sr^h, r^h tr^j, r^h tsr^j | 0 \leq h < 10, 0 \leq j < 5 \right\}. \tag{2.65}$$

Using Mathematica we decompose

$$\Phi_\Gamma = d^* M d \tag{2.66}$$

where M is the Hermitian coefficient matrix and d is the vector of monomials that appear in Φ_Γ. Again using Mathematica we find that M has rank 62 with 40 positive eigenvalues and 22 negative eigenvalues.

Remark 2.2.15. If $\Gamma < SU(2)$, and Γ is isomorphic to I, then $S(\Gamma) = (40, 22)$. Thus we have established part 5 of Theorem 1.1.1.

Chapter 3

The Asymptotic Positivity Ratio for Families of Subgroups of $U(2)$

The purpose of this chapter is to study the asymptotic behavior ratio of families of subgroups of $U(2)$. We use a combinatorial result from [15] to prove Theorem 1.1.2. Finally, we determine the asymptotic positivity ratio for dihedral subgroups, thereby proving Theorem 1.1.3.

3.1 Cyclic Groups

In this section we study the signs of the coefficients of the polynomial $f_{p,q}$. Since $\Gamma(p,q)$ is a diagonal subgroup, the sign of a coefficient of $f_{p,q}$ corresponds to the sign of an eigenvalue of the underlying matrix of $\Phi_{\Gamma(p,q)}$. When $q = 1$ or $q = 2$, we know the exact numbers of positive and negative coefficients. For general q however, it becomes difficult to determine these numbers exactly. Instead, we find upper and lower bounds for the number of coefficients of each sign. We use these bounds to compute the asymptotic positivity ratio as a rational function of q. Then we take the limit as q goes to infinity to show that for the $\Gamma(p,q)$ the asymptotic positivity ratio goes to $\frac{3}{4}$.

We first recall the *weight* of a polynomial from [4]. A polynomial $f(x,y)$ has weight j if $f(\omega x, \omega^q y) = \omega^{jp} f(x,y)$. In particular, a monomial $x^a y^b$ has weight j if $a + qb = jp$.

Suppose

$$f_{p,q} = \sum_{0 \le r,s \le p} \kappa_{r,s} x^r y^s. \tag{3.1}$$

Since $f_{p,q}$ is $\Gamma(p,q)$-invariant and the degree is at most p, we have $r + qs = kp$ for some $k \in \{1, \cdots, q\}$ and $r + s \le p$. In [3] D'Angelo shows that $\kappa_{r,s}$ is a non-zero integer whenever $x^r y^s$ is an invariant monomial, so the above question translates to determining the number of non-negative integer solutions to the equations $r + qs = kp$ for $k \in \{1, \cdots, q\}$ when $0 < r + s \le p$. For clarity we introduce the following notation.

Notation 3.1.1. The number of weight k terms in $f_{p,q}$ is $N_k(\Gamma(p,q))$. Denote the number of terms of odd weight by $N_{odd}(\Gamma(p,q))$ and the number of terms of even weight by $N_{even}(\Gamma(p,q))$.

Notice that the number of terms of $f_{p,q}$ is the same as the number of eigenvalues since we are using the diagonally generated cyclic group $\Gamma(p,q)$. The following two lemmas estimate the number of terms of each weight and the total number of terms.

Lemma 3.1.2. *The following inequality holds:*

$$\left|N_k(\Gamma(p,q)) - \frac{q-k}{q-1}\cdot\frac{p}{q}\right| \leq 1. \tag{3.2}$$

Proof. Fix p and q. We want to count the number of non-negative integer solutions (r,s) such that $r+qs=kp$ and $r+s\leq p$ where $1\leq k\leq q$. Notice that $r=kp-qs$, so r is an integer whenever s is an integer. Further notice that the two lines $r+qs=kp$ and $r+s=p$ intersect at the point $\left(\frac{p(q-k)}{q-1}, \frac{(k-1)p}{q-1}\right)$. Projecting onto the s coordinate, we observe that $N_k(p,q)$ is equal to the number of integers s such that $\frac{(k-1)p}{q-1}\leq s\leq\frac{kp}{q}$. Thus $N_k(\Gamma(p,q))$ is within 1 of $\left\lfloor\frac{kp}{q}-\frac{(k-1)p}{q-1}\right\rfloor=\left\lfloor\frac{q-k}{q-1}\cdot\frac{p}{q}\right\rfloor$. ☐

Remark 3.1.3. For $k=1$ the total number of solutions $N_1(\Gamma(p,q))$ is $\left\lfloor\frac{p}{q}\right\rfloor+1$.
For $k=q$,

$$N_q(\Gamma(p,q)) = 1. \tag{3.3}$$

Lemma 3.1.4. *The following inequality holds:*

$$\left|N(\Gamma(p,q)) - \frac{p}{2}\right| \leq q. \tag{3.4}$$

Proof. By Lemma 3.1.2,

$$\frac{q-k}{q-1}\cdot\frac{p}{q} - 1 \leq N_k(\Gamma(p,q)) \leq \frac{q-k}{q-1}\cdot\frac{p}{q}+1, \tag{3.5}$$

and by definition $N(\Gamma(p,q))=\sum_{k=1}^{q} N_k(\Gamma(p,q))$. Therefore applying Lemma 3.1.2 q times yields

$$\sum_{k=1}^{q}\frac{q-k}{q-1}\cdot\frac{p}{q} - q \leq N(\Gamma(p,q)) \leq \sum_{k=1}^{q}\frac{q-k}{q-1}\cdot\frac{p}{q}+q. \tag{3.6}$$

Factoring and rearranging the sum gives

$$\begin{aligned}\sum_{k=1}^{q}\frac{q-k}{q-1}\cdot\frac{p}{q} &= \frac{p}{q(q-1)}\sum_{k=1}^{q}(q-k)\\ &= \frac{p}{q(q-1)}(q^2-\frac{q(q+1)}{2}) = \frac{p}{2}.\end{aligned} \tag{3.7}$$

Thus combining the last two calculations gives the result

$$\frac{p}{2} - q \leq N(\Gamma(p,q)) \leq \frac{p}{2} + q. \tag{3.8}$$

□

Next we show in the limit that the ratio of the number of terms of odd weight to the total number of terms equals the ratio of the number of terms of even weight to the total number of terms.

Lemma 3.1.5. *The following limit holds:*

$$\lim_{q\to\infty}\lim_{p\to\infty} \frac{N_{odd}(\Gamma(p,q))}{N(\Gamma(p,q))} = \lim_{q\to\infty}\lim_{p\to\infty} \frac{N_{even}(\Gamma(p,q))}{N(\Gamma(p,q))} = \frac{1}{2} \tag{3.9}$$

Proof. There are four similar cases depending on the residue of q modulo 4. We consider the case where $q = 4r$. Recall

$$N_{odd}(\Gamma(p,q)) = \sum_{k=1}^{2r} N_{2k-1}(\Gamma(p,q)). \tag{3.10}$$

By Lemma 3.1.2

$$\sum_{k=1}^{2r} \frac{q-(2k-1)}{q-1} \cdot \frac{p}{q} - 2r \quad \leq \quad N_{odd} \leq \sum_{k=1}^{2r} \frac{q-(2k-1)}{q-1} \cdot \frac{p}{q} + 2r. \tag{3.11}$$

Rearranging and simplifying yields

$$\frac{p}{q-1} \cdot \frac{q}{4} - \frac{q}{2} \quad \leq \quad N_{odd} \leq \frac{p}{q-1} \cdot \frac{q}{4} + \frac{q}{2}. \tag{3.12}$$

Applying Lemma 3.1.4 gives

$$\frac{\frac{p}{q-1} \cdot \frac{q}{4} - \frac{q}{2}}{\frac{p}{2} + \frac{q}{2}} \leq \frac{N_{odd}}{N} \leq \frac{\frac{p}{q-1} \cdot \frac{q}{4} + \frac{q}{2}}{\frac{p}{2} - \frac{q}{2}}. \tag{3.13}$$

Simplfying gives

$$\frac{pq - 2q(q-1)}{2(p+q)(q-1)} \leq \frac{N_{odd}}{N} \leq \frac{pq + 2q(q-1)}{2(p-q)(q-1)}. \tag{3.14}$$

Taking the limit as $p \to \infty$ gives

$$\frac{q}{2(q-1)} \quad \leq \quad \lim_{p\to\infty} \frac{N_{odd}}{N} \leq \frac{q}{2(q-1)}. \tag{3.15}$$

Thus

$$\lim_{p\to\infty} \frac{N_{odd}}{N} = \frac{q}{2(q-1)}. \tag{3.16}$$

We proceed similarly for the even case. First

$$N_{even}(\Gamma(p,q)) = \sum_{k=1}^{2r} N_{2k}(\Gamma(p,q)). \tag{3.17}$$

Again we apply Lemma 3.1.2 to get

$$\sum_{k=1}^{2r} \frac{q-(2k)}{q-1} \cdot \frac{p}{q} - 2r \quad \leq \quad N_{even} \leq \sum_{k=1}^{2r} \frac{q-(2k)}{q-1} \cdot \frac{p}{q} + 2r. \tag{3.18}$$

Rearranging and simplifying yields

$$\frac{p}{q-1} \cdot \frac{q-2}{4} - \frac{q}{2} \quad \leq \quad N_{even} \leq \frac{p}{q-1} \cdot \frac{q-2}{4} + \frac{q}{2}. \tag{3.19}$$

Next apply Lemma 3.1.4 to get

$$\frac{\frac{p}{q-1} \cdot \frac{q-2}{4} - \frac{q}{2}}{\frac{p}{2} + \frac{q}{2}} \leq \frac{N_{even}}{N} \leq \frac{\frac{p}{q-1} \cdot \frac{q-2}{4} + \frac{q}{2}}{\frac{p}{2} - \frac{q}{2}}. \tag{3.20}$$

Simplifying gives

$$\frac{p(q-2) - 2q(q-1)}{2(p+q)(q-1)} \leq \frac{N_{even}}{N} \leq \frac{p(q-2) + 2q(q-1)}{2(p-q)(q-1)}. \tag{3.21}$$

Taking the limit as $p \to \infty$ gives

$$\frac{q-2}{2(q-1)} \quad \leq \quad \lim_{p\to\infty} \frac{N_{even}}{N} \leq \frac{q-2}{2(q-1)}. \tag{3.22}$$

Thus

$$\lim_{p\to\infty} \frac{N_{even}}{N} = \frac{q-2}{2(q-1)}. \tag{3.23}$$

While we have shown only one case, the others are similar; in Table 3.1 we summarize the cases.

Taking the limit as q goes to infinity in all cases gives the desired result that

$$\lim_{q\to\infty} \lim_{p\to\infty} \frac{N_{odd}}{N} = \frac{1}{2}. \tag{3.24}$$

Table 3.1: Summary

q	$\lim_p \frac{N_{even}}{N}$	$\lim_p \frac{N_{odd}}{N}$
$0 \pmod 4$	$\frac{q-2}{2(q-1)}$	$\frac{q}{2(q-1)}$
$1 \pmod 4$	$\frac{q-1}{2q}$	$\frac{q+1}{2q}$
$2 \pmod 4$	$\frac{q-2}{2(q-1)}$	$\frac{q}{2(q-1)}$
$3 \pmod 4$	$\frac{q-1}{2q}$	$\frac{q+1}{2q}$

Hence the limit the ratio of the number of even weight terms to the total number of terms equals the limit of the ratio of the number of odd weight terms to the total number of terms. □

So far we have ignored the signs. Using our notion of weight, we restate a theorem from [15].

Theorem 3.1.6. [Loehr, Warrington, Wilf] *The coefficient $\kappa_{r,s}$ of the weight w monomial $x^r y^s$ in $f_{p,q}$ is positive when $\gcd(r, s, w)$ is odd; the coefficient is negative when $\gcd(r, s, w)$ is even.*

Corollary 3.1.7. *The odd weight terms in $f_{p,q}$ are positive, and the even weight terms alternate sign.*

Proof. For odd weight terms $\gcd(r, s, w)$ is odd, thus by Theorem 3.1.6, the coefficients are positive.

When w is even, $\gcd(r, s, w)$ is even whenever both r and s are even. But $r = wp - qs$, so r is even if s and w are even. Finally for fixed weight w the possible s-values are consecutive integers. Hence for w even, s will alternate between even and odd values, thereby making $\gcd(r, s, w)$ alternate between even and odd values. By Theorem 3.1.6 the terms of odd weight in $f_{p,q}$ will alternate signs. □

For ease of notation we make the following definition.

Definition 3.1.8. Let $T(q)$ denote the asymptotic positivity ratio for $\Gamma(p, q)$, then

$$T(q) = \lim_{p\to\infty} L(\Gamma(p, q)). \tag{3.25}$$

The sequence $q \mapsto T(q)$ is of interest. We list its first few terms:

$$1,\ 1,\ \frac{5}{6},\ \frac{5}{6},\ \frac{4}{5},\ \frac{4}{5},\ \frac{11}{14},\ \frac{11}{14},\ \frac{7}{9}, \cdots . \tag{3.26}$$

Corollary 3.1.10 states that this sequence is monotone non-increasing, and each value repeats twice. Now we combine Theorem 3.1.6 with our previous estimates on the number of terms of each weight to compute the asymptotic positivity ratio.

Proposition 3.1.9. *The limit in the definition of the asymptotic positivity ratio exists, and*

$$T(q) = \begin{cases} \frac{3q+1}{4q} & \text{if } q \text{ is odd,} \\ \frac{3q-2}{4(q-1)} & \text{if } q \text{ is even.} \end{cases} \tag{3.27}$$

Proof. First consider the even and odd weights separately.

$$\begin{aligned} T(q) = \lim_{p\to\infty} \frac{N^{+}(\Gamma(p,q))}{N(\Gamma(p,q))} &= \lim_{p\to\infty} \frac{N^{+}_{odd}(\Gamma(p,q)) + N^{+}_{even}(\Gamma(p,q))}{N(\Gamma(p,q))} \\ &= \lim_{p\to\infty} \frac{N^{+}_{odd}(\Gamma(p,q))}{N(\Gamma(p,q))} + \lim_{p\to\infty} \frac{N^{+}_{even}(\Gamma(p,q))}{N(\Gamma(p,q))}. \end{aligned} \tag{3.28}$$

By Corollary 3.1.7 the odd weight terms are positive and the even weight terms alternate in sign. Hence $N_{odd} = N^{+}_{odd}$, and

$$\lim_{p\to\infty} \frac{N^{+}_{even}(\Gamma(p,q))}{N(\Gamma(p,q))} = \frac{1}{2} \cdot \lim_{p\to\infty} \frac{N_{even}(\Gamma(p,q))}{N(\Gamma(p,q))}. \tag{3.29}$$

When q is even, combining equations 3.28 and 3.29 with table 3.1 in Lemma 3.1.5 yields

$$T(q) = \lim_{p\to\infty} L(\Gamma(p,q)) = \frac{q}{2(q-1)} + \frac{1}{2} \cdot \frac{q-2}{2(q-1)} = \frac{3q-2}{4(q-1)}. \tag{3.30}$$

Similarly when q is odd, we have

$$T(q) = \lim_{p\to\infty} L(\Gamma(p,q)) = \frac{q+1}{2q} + \frac{1}{2} \cdot \frac{q-1}{2q} = \frac{3q+1}{4q}. \tag{3.31}$$

□

Corollary 3.1.10.

(i) The sequence $q \mapsto T(q)$ is monotone.

(ii) The limit as q goes to infinity of $T(q)$ exists.

(iii) $T(2r-1) = T(2r)$ for all $r \in \mathbb{Z}^{+}$.

Now taking the limit as q goes to infinity in Proposition 3.1.9 gives one of our main results.

Corollary 3.1.11. *Let $T(q)$ denote the asymptotic positivity ratio of $\Gamma(p,q)$. Then*

$$\lim_{q\to\infty} \lim_{p\to\infty} \frac{N^{+}(\Gamma(p,q))}{N(\Gamma(p,q))} = \lim_{q\to\infty} T(q) = \frac{3}{4}. \tag{3.32}$$

Notice that it also makes sense to consider this sequence as q goes to negative infinity. For instance, we already observed that the asymptotic positivity ratio for $\Gamma(p, p-1)$ is $\frac{1}{2}$. While it may not be immediately obvious, the previous results include this one as well. Since $\Gamma(p, p-1) = \Gamma(p, -1)$, we can plug $q = -1$ into our expression for $T(q)$. By 3.27, $T(-1) = \frac{1}{2}$. The case $q = 0$ is also perhaps interesting. We digress to consider it now:

$$\begin{aligned} f_{p,0} &= 1 - \prod_{j=0}^{p-1} \left(1 - \omega^j x - y\right) \\ &= 1 - (1-y)^p \prod_{j=0}^{p-1} \left(1 - \omega^j \frac{x}{(1-y)}\right) \\ &= 1 - (1-y)^p \left(1 - \frac{x^p}{(1-y)^p}\right) \\ &= 1 - (1-y)^p + x^p. \end{aligned} \tag{3.33}$$

Thus, $T(0) = \frac{1}{2}$, which again agrees with the result above.

3.2 Dihedral Group

The analysis can be extended to other groups in $U(2)$; for example, in this section, we define families of unitary representations of dihedral groups, and we determine the asymptotic positivity ratio to be $\frac{1}{2}$.

Let D_p denote the dihedral group with $2p$ elements; namely,

$$D_p := \left\langle a, b \mid a^p = b^2 = 1, bab = a^{-1} \right\rangle. \tag{3.34}$$

Without loss of generality, let $\iota : D_p \to U(2)$ be the faithful representation generated by

$$\iota(a) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \tag{3.35}$$

$$\iota(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.36}$$

Here the a corresponds to rotation, and the b corresponds to reflection. Let

$$\Delta_p = \iota(D_p). \tag{3.37}$$

Before stating the main results of this section we again begin with an example. We compute the number of positive and negative eigenvalues of $\Phi_{\Delta_3}(z,\overline{z})$. Expanding the product in the definition we get

$$\begin{aligned}\Phi_{\Delta_3} &= z_1^3\overline{z_1}^3 + z_2^3\overline{z_1}^3 - z_1^3z_2^3\overline{z_1}^6 + 6z_1z_2\overline{z_1}\,\overline{z_2} - 3z_1^4z_2\overline{z_1}^4\overline{z_2} - 3z_1z_2^4\overline{z_1}^4\overline{z_2} - 9z_1^2z_2^2\overline{z_1}^2\overline{z_2}^2 \\ &+ z_1^3\overline{z_2}^3 + z_2^3\overline{z_2}^3 - z_1^6\overline{z_1}^3\overline{z_2}^3 - z_2^6\overline{z_1}^3\overline{z_2}^3 - 3z_1^4z_2\overline{z_1}\,\overline{z_2}^4 - 3z_1z_2^4\overline{z_1}\,\overline{z_2}^4 - z_1^3z_2^3\overline{z_2}^6.\end{aligned} \tag{3.38}$$

In contrast to the cyclic case we get off-diagonal terms, and hence it is not enough to simply count the number of terms to get the number of eigenvalues. Rewriting in terms of a polynomials invariant under the D_3-action, we get

$$\begin{aligned}\Phi_{\Delta_3} &= (z_1^3+z_2^3)(\overline{z_1}^3+\overline{z_2}^3) - z_1^3z_2^3(\overline{z_1}^6+\overline{z_2}^6) - \overline{z_1}^3\overline{z_2}^3(z_1^6+z_2^6) + 6z_1z_2\overline{z_1}\,\overline{z_2} \\ &- 9z_1^2z_2^2\overline{z_1}^2\overline{z_2}^2 - 3z_1z_2(z_1^3+z_2^3)\overline{z_1}\,\overline{z_2}(\overline{z_1}^3+\overline{z_2}^3).\end{aligned} \tag{3.39}$$

Equivalently we get

$$\Phi_{\Delta_3} = \begin{pmatrix} \overline{z_1}^3+\overline{z_2}^3 \\ \overline{z_1}\,\overline{z_2}(\overline{z_1}^3+\overline{z_2}^3) \\ \overline{z_1}\,\overline{z_2} \\ \overline{z_1}^2\overline{z_2}^2 \\ \overline{z_1}^3\overline{z_2}^3 \\ \overline{z_1}^6+\overline{z_2}^6 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & -9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} z_1^3+z_2^3 \\ z_1z_2(z_1^3+z_2^3) \\ z_1z_2 \\ z_1^2z_2^2 \\ z_1^3z_2^3 \\ z_1^6+z_2^6 \end{pmatrix}. \tag{3.40}$$

Hence the eigenvalues of Φ_{Δ_3} are 1, -3, 6, -9, 1, -1. Then

$$S(\Delta_3) = (3,3). \tag{3.41}$$

For dihedral groups, we determine the asymptotic positivity ratio in the following theorem.

Theorem 3.2.1. *Let Δ_p be a dihedral group of order $2p$ in $U(2)$, then*

$$\lim_{p\to\infty} L(\Delta_p) = \frac{1}{2}. \tag{3.42}$$

Proof. In order to prove this theorem we first invoke Theorem 3.2.2 relating Φ_{Δ_p} to the more familiar $f_{p,p-1}$. In Lemma 3.2.3 below we count the number of positive and negative eigenvalues. Below in Corollary 3.2.4, we compute the positivity ratio, and we show that the limit as p goes to infinity exists. The conclusion of this theorem follows by taking the limit as p goes to infinity in Corollary 3.2.4. $\square$

D'Angelo [4] proves the following result relating Φ_{Δ_p} to $f_{p,p-1}$. The key idea is that the elements of Δ_p are either diagonal matrices or anti-diagonal matrices, and hence we can consider them by evaluating $f_{p,p-1}$ at different points.

Theorem 3.2.2. [4] *The invariant polynomial corresponding to the representation ι satisfies:*

$$\Phi_{\Delta_p} = f_{p,p-1}(|z_1|^2, |z_2|^2) + f_{p,p-1}(z_2\overline{z_1}, z_1\overline{z_2}) - f_{p,p-1}(|z_1|^2, |z_2|^2)f_{p,p-1}(z_2\overline{z_1}, z_1\overline{z_2}). \tag{3.43}$$

Unlike the general cyclic case, here we can exactly determine the numbers of positive and negative eigenvalues.

Lemma 3.2.3. *The total number of eigenvalues is*

$$N(\Delta_p) = p + \left\lfloor \frac{p}{2} \right\rfloor + 2. \tag{3.44}$$

The number of positive eigenvalues is

$$N^+(\Delta_p) = \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{p}{4} \right\rfloor + 2. \tag{3.45}$$

Proof. Recall

$$f_{p,p-1}(x,y) = x^p + y^p + \sum_{j=1}^{\lfloor \frac{p}{2} \rfloor} (-1)^j \kappa_j x^j y^j. \tag{3.46}$$

Let

$$B_p(x,y) = \sum_{j=1}^{\lfloor \frac{p}{2} \rfloor} (-1)^j \kappa_j x^j y^j. \tag{3.47}$$

We invoke Theorem 3.2.2 to decompose Φ_{Δ_p}; namely,

$$\begin{aligned}\Phi_{\Delta_p}(z,\overline{z}) = (z_1\overline{z_1})^p + (z_2\overline{z_2})^p + B_p(z_1\overline{z_1}, z_2\overline{z_2}) + (z_2\overline{z_1})^p + (z_1\overline{z_2})^p + B_p(z_2\overline{z_1}, z_1\overline{z_2})\\ - (z_1\overline{z_1})^p(f_{p,p-1}(z_2\overline{z_1}, z_1\overline{z_2})) - (z_2\overline{z_2})^p(f_{p,p-1}(z_2\overline{z_1}, z_1\overline{z_2}))\\ - (z_2\overline{z_1})^p B_p(z_1\overline{z_1}, z_2\overline{z_2}) - (z_1\overline{z_2})^p B_p(z_1\overline{z_1}, z_2\overline{z_2})\\ - B_p(z_1\overline{z_1}, z_2\overline{z_2})B_p(z_2\overline{z_1}, z_1\overline{z_2}).\end{aligned} \tag{3.48}$$

First notice

$$B_p(z_1\overline{z_1}, z_2\overline{z_2}) + B_p(z_2\overline{z_1}, z_1\overline{z_2}) = 2B_p(z_1\overline{z_1}, z_2\overline{z_2}). \tag{3.49}$$

Second we expand the last term in (3.48) to get

$$B_p(z_1\overline{z_1}, z_2\overline{z_2})B_p(z_2\overline{z_1}, z_1\overline{z_2}) = \sum_{k=2}^{2\left\lfloor \frac{p}{2} \right\rfloor} (-1)^k \sum_{\substack{a+b=k \\ 1\le a,b\le \left\lfloor \frac{p}{2} \right\rfloor}} \kappa_a\kappa_b(z_1\overline{z_1}z_2\overline{z_2})^k. \tag{3.50}$$

Define

$$E_k = \sum_{\substack{a+b=k \\ 1\le a,b\le \left\lfloor \frac{p}{2} \right\rfloor}} \kappa_a\kappa_b + 2\kappa_k \tag{3.51}$$

for $1 \le k \le \left\lfloor \frac{p}{2} \right\rfloor$, and

$$E_k = \sum_{\substack{a+b=k \\ 1\le a,b\le \left\lfloor \frac{p}{2} \right\rfloor}} \kappa_a\kappa_b \tag{3.52}$$

for $\left\lfloor \frac{p}{2} \right\rfloor < k \le 2\left\lfloor \frac{p}{2} \right\rfloor$.

Observe that

$$E_k > 0 \tag{3.53}$$

for all $1 \le k \le 2\left\lfloor \frac{p}{2} \right\rfloor$.

Now we want to write Φ_{Δ_p} in terms of invariant polynomials. The polynomials

$$z_1^p + z_2^p,\ z_1^j z_2^j,\ z_1^j z_2^j(z_1^p + z_2^p),\ z_1^{2p} + z_2^{2p} \tag{3.54}$$

are linearly independent and invariant under the D_p-action. Writing Φ_{Δ_p} in terms of these invariant polynomials we get

$$\begin{aligned}\Phi_{\Delta_p}(z,\overline{z}) \quad = \quad & (z_1^p + z_2^p)(\overline{z_1}^p + \overline{z_2}^p) + \sum_{k=1}^{2\left\lfloor \frac{p}{2} \right\rfloor} (-1)^{k+1} E_k (z_1 z_2 \overline{z_1 z_2})^k - z_1^p z_2^p(\overline{z_1}^{2p} + \overline{z_2}^{2p}) \\ & -\overline{z_1}^p\overline{z_2}^p(z_1^{2p} + z_2^{2p}) + \sum_{j=1}^{\left\lfloor \frac{p}{2} \right\rfloor} (-1)^j \kappa_j z_1^j z_2^j (z_1^p + z_2^p)\overline{z_1}^j\overline{z_2}^j(\overline{z_1}^p + \overline{z_2}^p).\end{aligned} \tag{3.55}$$

The underlying Hermitian matrix is nearly diagonal; we have only 2 non-diagonal terms. We now explicitly write out the polynomial in matrix form. Let

$$b = \begin{pmatrix} z_1^p + z_2^p \\ z_1^j z_2^j (z_1^p + z_2^p) \\ z_1^k z_2^k \\ z_1^{2p} + z_2^{2p} \end{pmatrix} \tag{3.56}$$

for $1 \leq j \leq \left\lfloor \frac{p}{2} \right\rfloor$ and $1 \leq k \leq p$. Also let

$$H_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A_{p,1} & 0 & 0 \\ 0 & 0 & A_{p,2} & 0 \\ 0 & 0 & 0 & A_{p,3} \end{pmatrix}. \tag{3.57}$$

The $\left\lfloor \frac{p}{2} \right\rfloor$ by $\left\lfloor \frac{p}{2} \right\rfloor$ diagonal matrix $A_{p,1}$ has diagonal entries given by

$$(A_{p,1})_{jj} = (-1)^j \kappa_j. \tag{3.58}$$

Next the matrix $A_{p,2}$ is p by p diagonal with diagonal entries given by

$$(A_{p,2})_{kk} = (-1)^{k+1} E_k. \tag{3.59}$$

Finally the matrix

$$A_{p,3} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{3.60}$$

when p is odd, and

$$A_{p,3} = \begin{pmatrix} (-1)E_p & -1 \\ -1 & 0 \end{pmatrix} \tag{3.61}$$

when p is even.

Thus

$$\Phi_{\Delta_p}(z, \overline{z}) = b^* H_p b. \tag{3.62}$$

Now we just count the eigenvalues of each diagonal submatrix. The matrix $A_{p,1}$ has $\left\lfloor \frac{p}{2} \right\rfloor$ eigenvalues and $\left\lfloor \frac{p}{4} \right\rfloor$ positive eigenvalues. The matrix $A_{p,2}$ has p eigenvalues and $\left\lfloor \frac{p}{2} \right\rfloor$ positive eigenvalues. In either case the matrix $A_{p,3}$ has 1 positive eigenvalue and 1 negative eigenvalue. Finally adding up the eigenvalues for each submatrix we get the desired result. $\square$

Corollary 3.2.4. *The following positivity ratios hold for the dihedral group:*

$$L(\Delta_p) = \begin{cases} \frac{1}{2} + \frac{2}{3p+4} & p \equiv 0 \bmod 4, \\ \frac{1}{2} + \frac{1}{3p+3} & p \equiv 1 \bmod 4, \\ \frac{1}{2} + \frac{1}{3p+4} & p \equiv 2 \bmod 4, \\ \frac{1}{2} & p \equiv 3 \bmod 4. \end{cases} \tag{3.63}$$

Moreover, the limit as p goes to infinity of the ratio equals $\frac{1}{2}$.

Proof. The four cases are similar. We consider only the case where $p \equiv 0 \bmod 4$. By Lemma 3.2.3 it follows that

$$N(\Delta_p) = p + \frac{p}{2} + 2 = \frac{3p+4}{2} \tag{3.64}$$

and

$$N^+(\Delta_p) = \frac{p}{2} + \frac{p}{4} + 2 = \frac{3p+8}{4}. \tag{3.65}$$

Hence the ratio is:

$$L(\Delta_p) = \frac{N^+(\Delta_p)}{N(\Delta_p)} = \frac{3p+8}{2(3p+4)} = \frac{1}{2} + \frac{2}{3p+4}. \tag{3.66}$$

$\square$

3.3 A Remark on Two-Dimensional Cyclic Signature Pairs

In this section we give an example of a cyclic group of order 21 with signature pair pair unequal to $S(\Gamma(21,q))$ for any q. Thus, in order to find all signature pairs for cyclic groups, one must study the more general class of diagonally generated cyclic groups. Let C_p be a cyclic group of order p, and let $\pi : C_p \to U(2)$ be a faithful group representation. Define $\Gamma_p = \pi(C_p)$ and let γ_p be a generator. It follows that γ_p is unitarily diagonalizable and thus there exists $U \in U(2)$ and $D = \mathrm{diag}(a_0, a_1)$ such that

$$\gamma_p = U^{-1} D U. \tag{3.67}$$

Since γ_p has order p, we know that $a_0^p = 1$ and $a_1^p = 1$. Thus

$$D = \begin{pmatrix} e^{2a\pi i/p} & 0 \\ 0 & e^{2b\pi i/p} \end{pmatrix} \tag{3.68}$$

where a and b are integers between 1 and p and $\gcd(p, a, b) = 1$. That is, $D = \Gamma(p; a, b)$.

Since the signature pair is preserved under conjugation,

$$S(\Gamma_p) = S(\Gamma(p; a, b)). \tag{3.69}$$

In order to find all possible signature pairs of Γ_p, it is therefore enough to study signature pairs of diagonally generated cyclic groups $\Gamma(p; a, b)$.

The goal now is to find the possible signature pairs $S(\Gamma(p; a, b))$ for diagonally generated cyclic groups. For example we list all signature pairs of Φ_{Γ_6}. By the previous theorem we need only consider the diagonally generated cyclic groups of order 6. The table below lists the signature pair in each case.

	(N^+, N^-)					
a	$b = 0$	1	2	3	4	5
0		(4,3)				(4,3)
1	(4,3)	(7,0)	(4,1)	(5,1)	(4,2)	(4,1)
2		(4,1)		(4,2)		(4,2)
3		(5,1)	(4,2)		(4,2)	(5,1)
4		(4,2)		(4,2)		(4,1)
5	(4,3)	(4,1)	(4,2)	(5,1)	(4,1)	(7,0)

The following signature pairs occur for Γ_6:

$$(7, 0),\ (4, 3),\ (5, 1),\ (4, 2),\ (4, 1).$$

Furthermore, notice that every possible signature pair occurs when $a = 1$. One naturally wonders whether this observation holds in general. Unfortunately, the answer is no. The first counterexample is the case $p = 21$, $a = 3$, and $b = 7$. (The next counterexample is $p = 30$, $a = 2$, and $b = 5$.)

We first list the distinct signature pairs for $\Gamma(21; 1, q)$ for $1 \leq q \leq 21$:

$$(22, 0), (12, 0), (11, 2), (9, 3), (12, 4),$$
$$(13, 2), (10, 3), (9, 4), (11, 4), (13, 3),$$
$$(7, 5), (12, 10).$$

However,

$$\begin{aligned} f_{21;3,7} = {} & 3x^7 - 3x^{14} + x^{21} + 7y^3 + 231x^7y^3 + 105x^{14}y^3 - 21y^6 + \\ & 1071x^7y^6 - 21x^{14}y^6 + 35y^9 + 798x^7y^9 - 35y^{12} + \\ & 84x^7y^{12} + 21y^{15} - 7y^{18} + y^{21}. \end{aligned} \tag{3.70}$$

Hence the signature pair of $\Gamma(21; 3, 7)$ is (11,5), which does not appear in the above list. Thus it is not enough to simply study the groups $\Gamma(p, q)$.

The reader may notice that $f_{21;1,7}$ and $f_{21;3,7}$ are superficially related; that is, they have the same coefficients up to sign. The other counterexamples do not resemble any $f_{p;1,q}$ so closely. For instance, $f_{30,2,5}$ is also a counterexample. In this case, none of the $f_{30,1,q}$ have the same coefficients as $f_{30,2,5}$.

Chapter 4

Number-Theoretic and Combinatorial Properties of CR Mappings

The polynomials $f_{p,q}$ have many interesting number-theoretic and combinatorial properties (see [4–6, 8, 11, 15–17]). We summarize some of these properties now. In [1] D'Angelo constructs the $f_{p,q}$ and shows that the coefficients are integers. In [4], he further proves that for each q, $f_{p,q}$ is congruent to $(x+y)^p$ mod (p) if and only if p is prime. The $f_{p,2}$ polynomials have an extremal property studied in [6]. Dilcher and Stolarsky consider a generalization of the $f_{p,2}$ polynomials in [8]. Osler uses a variant of $f_{p,2}$ to denest radicals in [17]. Musiker uses the $f_{p,2}$ polynomials while studying the combinatorics of elliptic curves [16]. Loehr, Warrington, and Wilf give a combinatorial interpretation for the coefficients of $f_{p,q}$ using circulant determinants [15]. They also gave a simple method of determining the sign of each term in $f_{p,q}$, which we used to calculate the asymptotic positivity ratio for the $\Gamma(p,q)$ in the previous chapter. They left open the problem of finding explicit formulas for the coefficients of $f_{p,q}$. In certain cases, we give explicit, simple formulas for these coefficients.

In [15], the authors also study the asymptotic behavior of the largest coefficient of $f_{p,q}$. D'Angelo uses methods from complex analysis to obtain more asymptotic information in [5]; for example, he gives an asymptotic formula for the sum of the coefficients of $f_{p,q}$. D'Angelo finds explicit recurrence relations for $f_{p,q}$ when $q \leq 3$ in [4]. In [11], Garnier and Ramaré study recurrence relations for $1 - f_{p,q}$; they show that $1 - f_{p,q}$ satisfies a linear recurrence relation of order at most 2^q. We proceed by using elementary symmetric functions and by studying each weight separately; this technique allows us to give explicit formulas for certain weight coefficients of $f_{p,q}$.

In this chapter, we study generating functions for the $f_{p,q}$. Using methods from [5], we show that these generating functions are rational of degree $2^q - 1$. As a corollary, we derive formulas for the coefficients in certain cases. Finally we recall from [15] a combinatorial interpretation of the coefficients of $f_{p,q}$ as the number of certain permutations. We also conjecture several possible generalizations to the methods from [15] and discuss the implications in CR geometry.

4.1 Generating Functions

In this section we first prove for fixed q that the sequence of polynomials $f_{p,q}$ satisfies a linear recurrence relation of order $2^q - 1$ and no smaller. Furthermore this sequence has a rational generating function in x, y, and the indeterminate. The method of proof appears in [5].

We now consider cyclic subgroups of $U(2)$. For the cyclic group C_p with p elements, we consider the group representations $\pi : C_p \to \Gamma(p; a, b) < U(2)$ generated by

$$s \mapsto \begin{pmatrix} \omega^a & 0 \\ 0 & \omega^b \end{pmatrix}, \tag{4.1}$$

where ω is a primitive p-th root of unity, $\gcd(a, b, p) = 1$, and s generates C_p. Without loss of generality, we assume $a \leq b$.

We define the polynomial $f_{p;a,b}$ by

$$f_{p;a,b}(|z_1|^2, |z_2|^2) = f_{p;a,b}(x, y) = 1 - \prod_{j=0}^{p-1} \left(1 - \omega^{aj} x - \omega^{bj} y\right) = \Phi_{\Gamma(p;a,b)}(z, \overline{z}). \tag{4.2}$$

The generating function for the $f_{p;a,b}$ is given by

$$\Psi(t, x, y) = \sum_{p=1}^{\infty} f_{p;a,b}(x, y) t^p. \tag{4.3}$$

The one-dimensional version of the $f_{p;a,b}$ plays crucial role in understanding the two-dimensional problem.

Lemma 4.1.1. *If ω is a p-th root of unity and a is a natural number, then*

$$1 - \prod_{j=1}^{p} \left(1 - \omega^{aj} x\right) = 1 - \left(1 - x^{\frac{p}{\gcd(p,a)}}\right)^{\gcd(p,a)}. \tag{4.4}$$

Proof. Suppose first that $\gcd(p, a) = 1$. The product

$$1 - \prod_{j=1}^{p} \left(1 - \omega^{aj} x\right) \tag{4.5}$$

is invariant under the transformation $x \mapsto \omega^a x$. Since x^p generates the algebra of invariant polynomials and the product has degree at most p, we know that

$$1 - \prod_{j=1}^{p} \left(1 - \omega^{aj} x\right) = a_0 + a_1 x^p \tag{4.6}$$

for some $a_0, a_1 \in \mathbb{C}$. Set $x = 0$ in the product to get $a_0 = 0$. Then set $x = 1$; the last factor in the product is zero and hence $a_1 = 1$.

Now suppose $\gcd(p, a) = g \neq 1$. The crucial observation is that

$$aj \cong a\left(j + \frac{p}{g}\right) \bmod p. \tag{4.7}$$

Thus each factor repeats g times in the product, so

$$1 - \prod_{j=1}^{p}\left(1 - \omega^{aj}x\right) = 1 - \left(\prod_{j=1}^{\frac{p}{g}}\left(1 - \omega^{aj}x\right)\right)^{g}. \tag{4.8}$$

Now $\gcd(a, \frac{p}{g}) = 1$ and ω^a is a $\frac{p}{g}$-th root of unity. Thus we apply the first case to complete the proof. □

Following [5], define the polynomial $g_{p;a,b}(t)$, expressed also in terms of the reciprocals of its roots, by

$$g_{p;a,b}(t) = 1 - xt^a - yt^b = \prod_{j=1}^{b}(1 - c_j(x, y)t). \tag{4.9}$$

Expressing $f_{p;a,b}$ in terms of $g_{p;a,b}$ gives

$$f_{p;a,b}(x, y) = 1 - \prod_{k=1}^{p} g_{p;a,b}(\omega^k). \tag{4.10}$$

Furthermore, let $A_{p,k}$ denote the sum of the weight k terms in $f_{p;a,b}$, and let Ψ_k denote the generating function given by

$$\Psi_k = \sum_{j=0}^{\infty} A_{j,k}t^j. \tag{4.11}$$

Part 1 of the following proposition appears in [5].

Theorem 4.1.2. *The following hold:*

1. *The polynomial $f_{p;a,b}$ is given by the following:*

$$f_{p;a,b} = \sum_{k=1}^{b}(-1)^{k+1} \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq k} (c_{j_1}c_{j_2}\cdots c_{j_k})^p. \tag{4.12}$$

2. The sum of the weight k terms in $f_{p;a,b}$ is given by

$$A_{p,k} = (-1)^{k+1} \sum_{1 \le j_1 < j_2 < \cdots < j_k \le k} (c_{j_1} c_{j_2} \cdots c_{j_k})^p. \tag{4.13}$$

Proof. The key idea is to factor g and use Lemma 4.1.1. Starting with Equation (4.10) and factoring g yields

$$\begin{aligned} f_{p:a,b} &= 1 - \prod_{k=0}^{p-1} g_{p;a,b}(\omega^k) \\ &= 1 - \prod_{k=0}^{p-1} \prod_{j=1}^{b} \left(1 - c_j \omega^k\right). \end{aligned} \tag{4.14}$$

Interchange the order of the products to get

$$f_{p;a,b} = 1 - \prod_{j=1}^{b} \prod_{k=0}^{p-1} \left(1 - c_j \omega^k\right). \tag{4.15}$$

Apply Lemma 4.1.1 to get

$$f_{p;a,b} = 1 - \prod_{j=1}^{b} \left(1 - c_j^p\right). \tag{4.16}$$

Thus, expanding the product completes the proof of part 1.

We now prove part 2. Recall a polynomial $h(x,y)$ is weight k with respect to $\Gamma(p;a,b)$ if $h(\lambda^a x, \lambda^b y) = \lambda^{kp} h(x,y)$. Notice that

$$g(\lambda^a x, \lambda^b y, t) = 1 - \lambda^a x t^a - \lambda^b y t^b = \prod_{j=1}^{b} \big(1 - c_j(\lambda^a x, \lambda^b y) t\big). \tag{4.17}$$

On the other hand,

$$g(\lambda^a x, \lambda^b y, t) = 1 - x\lambda^a t^a - y\lambda^b t^b = \prod_{j=1}^{b} (1 - c_j(x,y)\lambda t). \tag{4.18}$$

Therefore, up to reordering the roots, $c_i(\lambda^a x, \lambda^b y) = \lambda c_i(x,y)$. Let

$$G_k(x,y) = (-1)^{k+1} \sum_{1 \le j_1 < j_2 < \cdots < j_k \le k} (c_{j_1} c_{j_2} \cdots c_{j_k})^p. \tag{4.19}$$

Thus we get

$$G_k(\lambda^a x, \lambda^b y) = \lambda^{kp} G_k(x,y) \tag{4.20}$$

Therefore $G_k(x,y)$ has weight k, and hence $G_k = A_{p,k}$. $\square$

Corollary 4.1.3. *The following properties hold:*

1. *The $f_{p;a,b}$ themselves satisfy a linear recurrence relation of order $2^b - 1$ and no smaller.*

2. *The sum $A_{p,k}$ of the weight k terms of $f_{p;a,b}$ satisfies a linear recurrence relation of order $\binom{b}{k}$ and no smaller.*

Proof. Part 1 of Proposition 4.1.2 gives $f_{p;a,b}$ as the sum of 2^b-1 linear recurrences of order 1; it follows that the entire expression $f_{p;a,b}$ satisfies a linear recurrence relation of order $2^b - 1$. The coefficient of the degree $2^b - 1$ term is $(c_1 \cdots c_b)^{2^{b-2}} = ((-1)^{b-1}y)^{2^{b-2}}$. Since the coefficient of the highest order term is non-zero, we have that the recurrence relation cannot have order smaller than $2^b - 1$.

Similarly, part 2 of Proposition 4.1.2 expresses the weight k terms of $f_{p;a,b}$ as the sum of $\binom{b}{k}$ linear recurrences of order 1, and thus part 2 holds. □

Corollary 4.1.4. *The following properties hold:*

1. *The generating function Ψ is rational in t of degree $2^b - 1$ with coefficients in $\mathbb{Z}[x, y]$.*

2. *The generating function Ψ_k is rational in t of degree $\binom{b}{k}$ with coefficients in $\mathbb{Z}[x, y]$.*

Proof. Substituting (4.12) into (4.3) and summing the geometric series yields

$$\Psi(t, x, y) = \sum_{j=1}^{b} \frac{1}{1 - c_j t} - \sum_{1 \le i < j \le b} \left(\frac{1}{1 - c_i c_j t} \right) + \cdots + (-1)^{b+1} \frac{1}{1 - c_1 \cdots c_b t}. \tag{4.21}$$

Combining the fractions in (4.21) gives rational expression for Ψ where both the numerator and denominator are symmetric polynomials in the c_j's with integer coefficients. Given a symmetric polynomial with integer coefficients, we can write it in terms of the elementary symmetric polynomials also with integer coefficients. Since the elementary symmetric polynomials in c_j are polynomials in x and y with integer coefficients, Ψ is rational in x, y, and t. Also the denominator has degree $2^b - 1$. The proof of part 2 is similar. □

The previous result gives Ψ in terms of the reciprocals of the roots of the characteristic polynomial g. In general is it difficult to determine Ψ in terms x and y explicitly. In Section 4.2, we calculate Ψ in terms of x, y and t explicitly for $a = 1$ and $3 \le b \le 5$.

4.1.1 Formulas for Coefficients in Certain Cases

We give a general formula for the linear recurrence relation for the weight 1 and weight $q - 1$ terms. Using this recurrence relation, we find formulas for the coefficients of $f_{p;a,b}$ in these cases. The terms of weight q

in $f_{p,q}$ are particularly easy to understand:

$$A_{p,q} = (-1)^q y^p. \tag{4.22}$$

This fact follows either from Proposition 4.1.2 and $\prod_{j=1}^q c_j = (-1)^{q-1} y$. One can also use D'Angelo's Uniqueness Theorem from Chapter 1 to see this fact.

By Proposition 4.1.2, the weight 1 terms have particularly simple expression in terms of the c_j

$$A_{p,1} = c_1^p + c_2^p + \cdots + c_q^p, \tag{4.23}$$

but the formula in terms of x and y is more complicated. We therefore obtain formulas involving x and y by using recurrence relations.

The following theorem is proven for $q \leq 3$ in [4].

Theorem 4.1.5. (Recurrence Relations)
(1) The weight one terms of $f_{p,q}$ satisfy the q-th order recurrence

$$A_{p+q,1} = xA_{p+q-1,1} + yA_{p,1} \tag{4.24}$$

with initial conditions

$$\begin{aligned} A_{1,1} &= x && (4.25)\\ A_{2,1} &= x^2 && (4.26)\\ &\vdots \\ A_{q-1,1} &= x^{q-1} && (4.27)\\ A_{q,1} &= x^q + qy && (4.28) \end{aligned}$$

(2) The weight $q-1$ terms of $f_{p,q}$ satisfy the q-th order recurrence

$$A_{p+q,q-1} = (-1)^{q-2} xy^{q-2} A_{p+1,q-1} + y^{q-1} A_{p,q-1} \tag{4.29}$$

with initial conditions

$$
\begin{aligned}
A_{1,q-1} &= 0 && (4.30)\\
A_{2,q-1} &= 0 && (4.31)\\
&\vdots\\
A_{q-2,q-1} &= 0 && (4.32)\\
A_{q-1,q-1} &= (q-1)xy^{q-2} && (4.33)\\
A_{q,q-1} &= (-1)^q q y^{q-1}. && (4.34)
\end{aligned}
$$

Proof. We prove (1); the proof of part (2) is similar. One can easily verify the initial conditions; we thus omit the details of that calculation. Let e_n denote the n-th elementary symmetric function in $c_1, \cdots, c_q$. Recall

$$1 - xt - yt^q = \prod_{j=1}^{q} (1 - c_j(x,y)t). \tag{4.35}$$

Therefore,

$$e_n = \begin{cases} x & \text{if } n = 1, \\ 0 & \text{if } 2 \leq n < q, \\ (-1)^{q-1}y & \text{if } n = q, \\ 0 & \text{if } n > q. \end{cases} \tag{4.36}$$

By Newton's Theorem on symmetric polynomials,

$$A_{n,1} = e_1 A_{n-1,1} - e_2 A_{n-2,1} + e_3 A_{n-3,1} - \cdots + (-1)^{n-2} e_{n-1} A_{1,1} - n e_n. \tag{4.37}$$

Plug in the equations from 4.36 to get

$$A_{n,1} = x A_{n-1,1} + y A_{n-q,1}. \tag{4.38}$$

□

We include the following corollary of Proposition 4.1.2 in the case of $a = 1$ and $b = q$.

Corollary 4.1.6. *The weight w terms of $f_{p,q}$ satisfy a recurrence relation of order $\binom{q}{w}$. Moreover, the* *recurrence relation satisfies*

$$A_{p,w} = \sum_{k=1}^{\binom{q}{w}} a_k x^{wk - \lfloor \frac{wk}{q} \rfloor \cdot q} y^{\lfloor \frac{wk}{q} \rfloor} A_{p-k,w} \tag{4.39}$$

for some $a_k \in \mathbb{Z}$.

Next we use the recurrence relations to get the generating functions for these sequences of polynomials.

Proposition 4.1.7. *The generating functions in the weight 1 and weight* $q-1$ *cases are give by the following.*
(1) For the weight one terms, the generating function is given by

$$\Psi_1(x, y, t) = \frac{q - (q-1)xt}{1 - xt - yt^q}. \tag{4.40}$$

(2) For the weight $q - 1$ *terms, the generating function is given by*

$$\Psi_{q-1}(x, y, t) = \frac{(-1)^q q - xy^{q-2}t^{q-1}}{1 + (-1)^{q-1}xy^{q-2}t^{q-1} - y^{q-1}t^q}. \tag{4.41}$$

Proof. We first prove part (1). Define $A_{0,1} = q$ and $A_{0,q-1} = (-1)^q q$. Theorem 4.1.5 gives

$$\sum_{n=q}^{\infty} A_{n,1}t^n = x \sum_{n=q}^{\infty} A_{n-1,1}t^n + y \sum_{n=q}^{\infty} A_{n-q,1}t^n \tag{4.42}$$

$$\Psi_1(x, y, t) - \sum_{n=0}^{q-1} A_n t^n = xt \left(\Psi_1(x, y, t) - \sum_{n=0}^{q-2} A_{n,1}t^n \right) + yt^q \Psi_1(x, y, t). \tag{4.43}$$

Rearranging the last equation gives

$$\Psi_1(x, y, t)(1 - xt - yt^q) = \sum_{n=0}^{q-1} A_n t^n - xt \sum_{n=0}^{q-2} A_{n,1}t^n \tag{4.44}$$

$$= (A_{0,1}) + (A_{1,1} - xA_{0,1})t + \ldots + (A_{q-2,1} - xA_{q-3,1})t^{q-2} + (A_{q-1,1} - xA_{q-2,1})t^{q-1}. \tag{4.45}$$

After substituting the initial values in the last equation we get

$$\Psi_1(x, y, t)(1 - xt - yt^q) = q + (x - qx)t, \tag{4.46}$$

and therefore (4.40) holds.

The proof for the $q - 1$ case is largely the same as in the weight 1 case. Again we use the recurrence relation to get the generating function:

$$\sum_{n=q}^{\infty} A_{n,q-1}t^n = (-1)^{q-2}xy^{q-2}\sum_{n=q}^{\infty} A_{n-q+1,q-1}t^n + y^{q-1}\sum_{n=q}^{\infty} A_{n-q,q-1}t^n. \tag{4.47}$$

Rearranging we get

$$\Psi_{q-1}(x,y,t)(1-(-1)^{q-2}xy^{q-2}t^{q-1} - y^{q-1}t^q) = \sum_{n=0}^{q-1} A_{n,q-1}t^n - (-1)^{q-2}xy^{q-2}t^{q-1}(A_{0,q-1}). \tag{4.48}$$

Substituting for initial values and solving for $\Psi_{q-1}(x,y,t)$ gives the desired result. □

Using the generating functions we can find simple formulas for the coefficients of the $f_{p,q}$ of particular weights. See [2] for more formulas for the case $f_{p,2}$.

Theorem 4.1.8. *Let $\kappa_{r,s}(p,q)$ be the coefficient of $x^r y^s$ in $f_{p,q}$. 1. The coefficients of the weight one polynomials are given by*

$$\kappa_{r,s}(p,q) = q\binom{p-(q-1)s-1}{p-qs} + \binom{p-(q-1)s-1}{p-qs-1} = \frac{p}{s}\binom{p-(q-1)s-1}{s-1}. \tag{4.49}$$

2. The coefficients of the weight $q-1$ polynomials are given by

$$\kappa_{r,s}(p,q) = (-1)^{q(M+1)}q\binom{p-s}{M} + (-1)^{q(M+1)+1}\binom{p-s-1}{M-1} \tag{4.50}$$

where $M = (q-1)p - qs$.

Proof. We start with the generating function, rewrite it as a geometric series, expand it using the binomial theorem, and pick out the coefficient of $x^{p-qs}y^s t^p$:

$$\begin{aligned}
\Psi_1(x,y,t) &= \frac{q+(q-1)xt}{1-xt-yt^q} \\
&= (q+(q-1)xt)\sum_{i=0}^{\infty}(xt+yt^q)^i \\
&= (q+(q-1)xt)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}(xt)^j(yt^q)^{i-j} \\
&= q\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}x^j y^{i-j}t^{q(i-j)+j} + (q-1)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}x^{j+1}y^{i-j}t^{q(i-j)+1+j}.
\end{aligned} \tag{4.51}$$

We extract the coefficient of $x^{p-qs}y^s t^p$ in both sums to get the desired result. The proof again is similar to the previous case. Starting with the generating function, we expand as follows:

$$
\begin{aligned}
\Psi_{q-1}(x,y,t) &= \frac{(-1)^q q - xy^{q-2}t^{q-1}}{1+(-1)^{q-1}xy^{q-2}z^{q-1} - y^{q-1}z^q} \\
&= ((-1)^q q - xy^{q-2}t^{q-1}) \sum_{i=0}^{\infty} ((-1)^q xy^{q-2}t^{q-1} + y^{q-1}t^q)^i \\
&= ((-1)^q q - xy^{q-2}t^{q-1}) \sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{i}{j} ((-1)^q xy^{q-2}t^{q-1})^j (y^{q-1}t^q)^{i-j} \\
&= q \sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{i}{j} (-1)^{q(j+1)} x^j y^{i(q-1)-j} t^{qi-j} \\
&+ \sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{i}{j} (-1)^{qj+1} x^{j+1} y^{(i+1)(q-1)-j-1} t^{q(i+1)-j-1}.
\end{aligned} \tag{4.52}
$$

The coefficient of $x^r y^s t^p$ in the first sum is

$$
(-1)^{q((q-1)p-qs+1)} q \binom{p-s}{(q-1)p-qs}, \tag{4.53}
$$

and the coefficient in the second sum is

$$
(-1)^{q(q(p-s)-p-1)+1} \binom{p-s-1}{(q-1)p-qs-1}. \tag{4.54}
$$

Adding (4.53) and (4.54) completes the proof of part (2). □

We can use the other generating functions to find similar formulas for the coefficients of $f_{p,q}$ in terms of binomial coefficients.

In [3], D'Angelo shows that $f_{p,2}$ and $f_{p,p-1}$ have the same coefficients up to sign. We now give a modest extension of this result to conclude this section. We show that $f_{p,q}$ and $f_{p,p-q+1}$ have the same coefficients up to sign. The proofs are essentially the same.

Proposition 4.1.9. *The polynomials $f_{p,q}$ and $f_{p,p-q+1}$ satisfy the following identity:*

$$
f_{p,q}(x,y) = 1 + x^p \left(1 - f_{p,p-q+1}\left(\frac{1}{x}, \frac{-y}{x}\right)\right). \tag{4.55}
$$

Proof. Recall the definition of $f_{p,q}$ below:

$$
f_{p,q}(x,y) = 1 - \prod_{j=0}^{p-1} \left(1 - x\omega^j - y\omega^{qj}\right). \tag{4.56}
$$

We factor and re-index:

$$\begin{aligned}
f_{p,q}(x,y) &= 1 - \prod_{j=0}^{p-1} \left(-x\omega^j\right) \left(-\frac{1}{x}\omega^{-j} + 1 + \frac{y}{x}\omega^{(q-1)j}\right) \\
&= 1 - (-1)^p \prod_{j=1}^{p} \omega^j x^p \prod_{j=0}^{p-1} \left(1 - \frac{1}{x}\omega^j - \frac{-y}{x}\omega^{-(q-1)j}\right). \qquad (4.57)
\end{aligned}$$

Notice that

$$\prod_{j=1}^{p} \omega^j = (-1)^{p+1}. \qquad (4.58)$$

Simplifying completes the proof:

$$\begin{aligned}
f_{p,q}(x,y) &= 1 + x^p \prod_{j=0}^{p-1} \left(1 - \frac{1}{x}\omega^j - \frac{-y}{x}\omega^{p-(q-1)j}\right) \\
&= 1 + x^p \left(1 - f_{p,p-q+1}\left(\frac{1}{x}, \frac{-y}{x}\right)\right). \qquad (4.59)
\end{aligned}$$

□

Corollary 4.1.10. *The polynomials $f_{p,q}$ and $f_{p,p-q+1}$ have the same coefficients up to sign.*

Proof. The previous proposition gives

$$f_{p,q}(x,y) = 1 + x^p \left(1 - f_{p,p-q+1}\left(\frac{1}{x}, \frac{-y}{x}\right)\right). \qquad (4.60)$$

The coefficients of $f_{p,p-q+1}$ are obviously unchanged by the change of variables. Recall that

$$f_{p,p-q+1}\left(\frac{1}{x}, \frac{-y}{x}\right) = \left(\frac{1}{x}\right)^p + (-1)^p \left(\frac{y}{x}\right)^p + \text{mixed}. \qquad (4.61)$$

Obviously the mixed terms have at most a sign change, and the leading 1 will cancel with the $\frac{1}{x^p}$ term. Thus the coefficients are the same up to a sign change. □

4.2 Explicit Recurrence Relations for $q \leq 4$

In this section, we give an alternate approach to the generating functions of $f_{p;a,b}$. We define a sequence of polynomials, and we show that they satisfy the 4 properties of D'Angelo's Uniqueness Theorem (see Chapter 1 and [3]). This approach was used by D'Angelo in [4] to find recurrence relations for each weight in the $f_{p;a,b}$ when $a = 1$ and $1 \leq b \leq 3$. We also give the explicit recurrence relation for $f_{p;1,q}$ for $q = 4$. We recall the $q = 3$ case from [4], and we prove the $q = 4$ case using the ideas from [4].

Theorem 4.2.1. [q=3] *The individual weight polynomials satisfy the following recurrences.*

$$A_{p+3,1} = xA_{p+2,1} + yA_{p,1} \tag{4.62}$$

$$A_{p+3,2} = -xyA_{p+1,2} + y^2A_{p,2} \tag{4.63}$$

$$A_{p+1,3} = yA_{p,3} \tag{4.64}$$

Combining these recurrence relations together gives the following 7-th order recurrence for the $f_{p,3}$.

$$\begin{aligned} f_{p+7,3} &= (x+y)f_{p+6,3} - 2xyf_{p+5,3} + (y + y^2 + x^2y + xy^2)f_{p+4,3} \\ &- (y^2 + xy^2 + x^2y^2 + y^3)f_{p+3,3} + (xy^2 + xy^3)f_{p+2,3} - (y^3 + xy^3)f_{p+1,3} + y^4 f_{p,3} \end{aligned} \tag{4.65}$$

with initial conditions

$$f_{1,3} = x + y \tag{4.66}$$

$$f_{2,3} = x^2 + 2xy + y^2 \tag{4.67}$$

$$f_{3,3} = x^3 + 3y - 3y^2 + y^3 \tag{4.68}$$

$$f_{4,3} = x^4 + 4xy - 2x^2y^2 + y^4 \tag{4.69}$$

$$f_{5,3} = x^5 + 5x^2y + 5xy^3 + y^5 \tag{4.70}$$

$$f_{6,3} = x^6 + 6x^3y + 3y^2 + 2x^3y^3 - 3y^4 + y^6 \tag{4.71}$$

$$f_{7,3} = x^7 + 7x^4y + 7xy^2 - 7x^2y^4 + y^7. \tag{4.72}$$

Proposition 4.2.2. [q=4] *The weight k terms of $f_{p,4}$ satisfy the $\binom{4}{k}$-th order recurrences given below:*

$$A_{p+4,1} = xA_{p+3,1} + yA_{p,1}, \tag{4.73}$$

$$A_{p+6,2} = -yA_{p+4,2} - x^2yA_{p+3,2} + y^2A_{p+2,2} + y^3A_{p,2}, \tag{4.74}$$

$$A_{p+4,3} = xy^2A_{p+1,3} + y^3A_{p,3}, \tag{4.75}$$

$$A_{p+1,4} = -yA_{p,4}, \tag{4.76}$$

with initial conditions taken from the weight k terms among the first $\binom{4}{k}$ $f_{p,4}$. Combining all these together we get a recurrence of order $2^4-1=15$ for $f_{p,4}$ given by

$$\begin{aligned}
f_{p+15,4} &= (x-y)f_{p+14,4} + (-y+xy)f_{p+13,4} + (xy - x^2y - y^2 + xy^2)f_{p+12,4} \\
&\quad + (y + x^3y + y^2 + xy^2 - 2x^2y^2 + y^3 + xy^3)f_{p+11,4} \\
&\quad + (y^2 - xy^2 + x^3y^2 + y^3 - x^2y^3 + y^4)f_{p+10,4} \\
&\quad + (y^2 + y^3 - xy^3 - x^2y^3 + x^3y^3 + y^4)f_{p+9,4} \\
&\quad + (x^2y^2 + y^3 - 2xy^3 - x^4y^3 + y^4 - 2xy^4 + x^3y^4 + y^5)f_{p+8,4} \\
&\quad + (-y^3 + x^2y^3 - y^4 - 2xy^4 + x^2y^4 - x^3y^4 - x^4y^4 - y^5 - 2xy^5 + x^2y^5)f_{p+7,4} \\
&\quad + (-y^4 - xy^4 - y^5 + x^2y^5 - x^3y^5 - y^6)f_{p+6,4} \\
&\quad + (-y^4 - x^3y^4 - y^5 - xy^5 + x^2y^5 - y^6)f_{p+5,4} \\
&\quad + (-y^5 + xy^5 - x^2y^5 - x^3y^5 - y^6 + xy^6 + x^2y^6 - y^7)f_{p+4,4} \\
&\quad + (y^6 + xy^6 - x^2y^6 + xy^7)f_{p+3,4} + (xy^6 + y^7)f_{p+2,4} + (y^7 + xy^7)f_{p+1,4} + y^8 f_{p,4}
\end{aligned} \tag{4.77}$$

with initial conditions

$$\begin{aligned}
f_{1,4} &= x + y \\
f_{2,4} &= x^2 + 2y - y^2 \\
f_{3,4} &= x^3 + 3x^2y + 3xy^2 + y^3 \\
f_{4,4} &= x^4 + 4y - 6y^2 + 4y^3 - y^4 \\
f_{5,4} &= x^5 + 5xy - 5x^2y^2 + y^5 \\
f_{6,4} &= x^6 + 6x^2y - 3x^4y^2 + 2y^3 + 3x^2y^4 - y^6 \\
f_{7,4} &= x^7 + 7x^3y + 14x^2y^3 + 7xy^5 + y^7 \\
f_{8,4} &= x^8 + 8x^4y + 4y^2 + 8x^4y^3 - 6y^4 + 4y^6 - y^8 \\
f_{9,4} &= x^9 + 9x^5y + 9xy^2 + 3x^6y^3 - 18x^2y^4 + 3x^3y^6 + y^9 \\
f_{10,4} &= x^{10} + 10x^6y + 15x^2y^2 - 25x^4y^4 + 2y^5 + 10x^2y^7 - y^{10} \\
f_{11,4} &= x^{11} + 11x^7y + 22x^3y^2 - 11x^6y^4 + 33x^2y^5 + 11xy^8 + y^{11} \\
f_{12,4} &= x^{12} + 12x^8y + 30x^4y^2 + 4y^3 - 3x^8y^4 + 48x^4y^5 - 6y^6 + 3x^4y^8 + 4y^9 - y^{12} \\
f_{13,4} &= x^{13} + 13x^9y + 39x^5y^2 + 13xy^3 + 39x^6y^5 - 39x^2y^6 + 13x^3y^9 + y^{13} \\
f_{14,4} &= x^{14} + 14x^{10}y + 49x^6y^2 + 28x^2y^3 + 14x^8y^5 - 98x^4y^6 + 2y^7 + 21x^2y^{10} - y^{14} \\
f_{15,4} &= x^{15} + 15x^{11}y + 60x^7y^2 + 50x^3y^3 + 3x^{10}y^5 - 95x^6y^6 + 60x^2y^7 + 3x^5y^{10} + 15xy^{11} + y^{15}.
\end{aligned}$$

Proof. It is tedious but straightforward to check the initial conditions, so we omit the details. We prove the polynomials defined by

$$q_p(x, y) = A_{p,1} + A_{p,2} + A_{p,3} + A_{p,4} \tag{4.78}$$

satisfy the four conditions of D'Angelo's Uniqueness Theorem, thereby giving $q_p = f_{p,4}$ and the recurrence relation for $f_{p,4}$. Conditions 1 and 3 are clear. From here on, we assume $p > 15$.

Now we verify condition 2. Using Mathematica and the 15-th order recurrence relation for $q_p(x, y)$, we calculate $q_p(1 - y, y) = 1$. Thus q_p is identically 1 on the line $x + y = 1$.

Again we must verify condition 4 for each weight individually. For weight 1,

$$\begin{aligned} A_{p,1}(\lambda x, \lambda^4 y) &= \lambda x A_{p-1,1}(\lambda x, \lambda^4 y) + \lambda^4 y A_{p-4,1}(\lambda x, \lambda^4 y) \\ &= \lambda^p (x A_{p-1,1}(x, y) + y A_{p-4,1}(x, y)) = \lambda^p A_{p,1}(x, y). \end{aligned} \tag{4.79}$$

Choose λ to be a p-th root of unity, then $A_{p,1}$ is $\Gamma(p, 4)$-invariant. For weight 2,

$$\begin{aligned} A_{p,2}(\lambda x, \lambda^4 y) &= -\lambda^4 y A_{p-2,2}(\lambda x, \lambda^4 y) - \lambda^6 x^2 y A_{p-3,2}(\lambda x, \lambda^4 y) \\ &+ \lambda^8 y^2 A_{p-4,2}(\lambda x, \lambda^4 y) + \lambda^{12} y^3 A_{p-6,2} \\ &= \lambda^p (-y A_{p-2,2} - x^2 y A_{p-3,2} + y^2 A_{p-4,2} + y^3 A_{p-6,2}) = \lambda^p A_{p,2}(x, y). \end{aligned} \tag{4.80}$$

Choose λ to be a root of unity of order p, then $A_{p,2}$ is $\Gamma(p, 4)$-invariant. The other weights are similar, so we omit the proofs. Since q_p is a sum of $\Gamma(p, 4)$-invariant polynomials, g_p must be invariant. For fixed p, the polynomial q_p satisfies the four conditions of D'Angelo's Uniqueness Theorem, and hence $q_p = f_{p,4}$. □

4.3 Combinatorial Interpretation of the Coefficients

In this section, we recall the combinatorial aspects of the coefficients of the $f_{p,q}$ given in [15].

Remark 4.3.1. The authors in [15] actually study the polynomial given by $1 - f_{p;a,b}$. Thus one needs to change the signs of the coefficients to pass from our notation to theirs.

We now describe the theorem and a possible generalization. We expand the product in (4.2) and write

$$f_{p;a,b} = \sum_{r+s \leq p} \kappa_{r,s} x^r y^s. \tag{4.81}$$

In the theorem from [15], the authors determine the sign of the coefficients of $f_{p;a,b}$ for certain values of a

and b. We propose a modest generalization now:

Conjecture 4.3.2. *If* $\gcd(p, a, b) = 1$, *then*

(0) $\kappa_{r,s} = 0$ *if* p *does not divide* $ar + bs$ *or if* $r = s = 0$.

Furthermore, if p *does divide* $ar + bs$, *then*

1. $\kappa_{r,s} < 0$ *if* $\gcd(r, s, \frac{ar+bs}{p})$ *is even, and*

2. $\kappa_{r,s} > 0$ *if* $\gcd(r, s, \frac{ar+bs}{p})$ *is odd.*

In [15], the result is shown in the cases where $\gcd(p, a) = 1$ or $\gcd(p, b) = 1$. In order to find all the signature pairs for cyclic subgroups of $U(2)$, we need to study the more general case where $\gcd(p, a, b) = 1$. The method of proof should be largely the same as in [15], but there are many details to check. The basic idea for the proof is to represent $1 - f_{p;a,b}$ as a circulant determinant, allowing one to give a combinatorial interpretation of the coefficients. The coefficients are then given by the cardinality of a set of a certain type of permutations. Finally, one constructs such permutations to show which coefficients are non-zero. Furthermore, the cycle type of the permutations determines the sign of the coefficient.

Now we discuss a few examples. The first example not included in their original theorem is the case: $p = 6$, $a = 2$, and $b = 3$. Expanding the product in (4.2) gives

$$f_{6;2,3} = 2x^3 - x^6 + 3y^2 + 6x^3y^2 - 3y^4 + y^6. \tag{4.82}$$

In this case, the sign of the monomial $\kappa_{r,s}x^ry^s$ is determined by the parity of $\gcd(r, s, \frac{ar+bs}{p})$.

In the $(6; 2, 4)$ case, the polynomial is given by

$$f_{6,2,4} = 2x^3 - x^6 + 6xy - 6x^4y - 9x^2y^2 + 2y^3 - 2x^3y^3 - 6xy^4 - y^6. \tag{4.83}$$

The monomials $-6xy^4$ and $-2x^3y^3$ have negative coefficients, and the $\gcd(r, s, \frac{2r+4s}{6})$ is odd in both cases. Hence the hypothesis $\gcd(p, a, b) = 1$ is necessary in Conjecture 4.3.2. Moreoever, if $\gcd(p, a, b) = m \neq 1$, then we express $f_{p;a,b}$ in terms of $f_{\frac{p}{m};\frac{a}{m},\frac{b}{m}}$.

Proposition 4.3.3. *If* $\gcd(p, a, b) = m$, *then*

$$f_{p;a,b} = 1 - \left(1 - f_{\frac{p}{m};\frac{a}{m},\frac{b}{m}}\right)^m. \tag{4.84}$$

Proof. Let ω be a primitive p-th root of unity. If p, a, and b have a common factor m, then the matrix

$$\begin{pmatrix} \omega^a & 0 \\ 0 & \omega^b \end{pmatrix}$$

does not generate a cyclic group of order p. In fact,

$$\begin{pmatrix} \omega^a & 0 \\ 0 & \omega^b \end{pmatrix}^j = \begin{pmatrix} \omega^a & 0 \\ 0 & \omega^b \end{pmatrix}^{\frac{p}{m}+j} \tag{4.85}$$

Thus the following product

$$\prod_{j=1}^{p} \left(1 - \omega^{aj}x - \omega^{bj}y\right) \tag{4.86}$$

has repeated factors. That is,

$$\prod_{j=1}^{p} \left(1 - \omega^{aj}x - \omega^{bj}y\right) = \prod_{j=1}^{\frac{p}{m}} \left(1 - \omega^{aj}x - \omega^{bj}y\right)^m = \prod_{j=1}^{\frac{p}{m}} \left(1 - \eta^{\frac{a}{g}j}x - \eta^{\frac{b}{g}j}y\right)^m \tag{4.87}$$

where $\eta = \omega^m$. Since η is a root of unity of order $\frac{p}{m}$, (4.84) holds. $\square$

For many problems, it suffices to study the case when $\gcd(p, a, b) = 1$. In particular, when studying signature pairs of cyclic groups, it is natural to restrict to the case where $\gcd(p, a, b) = 1$.

Corollary 4.3.4. *For* $\gcd(p, a, b) = 1$*,*

$$\lim_{a\to\infty} \lim_{b\to\infty} \lim_{p\to\infty} \frac{N^+(\Gamma(p; a, b)}{N(\Gamma(p; a, b)} = \frac{3}{4}. \tag{4.88}$$

Figure 1 shows the pattern of the signs of the coefficients in the $(ab; a, b)$ case when $\gcd(a, b) = 1$. The figure suggests why one expects the ratio of the number of positive terms to the total number of terms to roughly equal $\frac{3}{4}$ when a, b, and p are large.

4.3.1 Formulas for the Coefficients

Before we examine the proof from [15], we determine the coefficients of $f_{p;a,b}$ in certain cases. We use the one-dimensional case to get formulas for the pure terms in $f_{p;a,b}$.

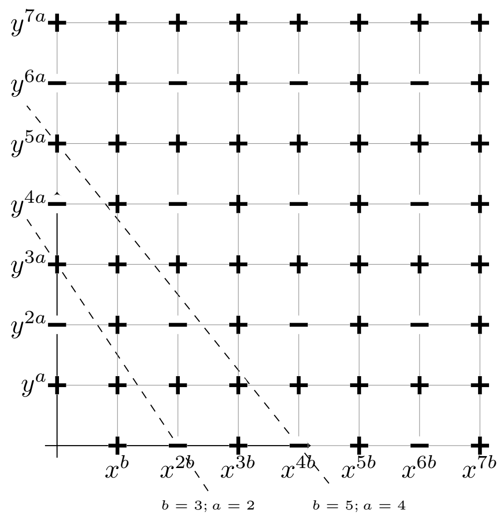

Figure 4.1: Signs of the coefficients of $f_{ab;a,b}$. The signs below the dashed line represent the signs of the monomials that appear in $f_{ab;a,b}$ for the particular values of a and b.

Proposition 4.3.5. *Let* $g_1 = \gcd(p,a)$ *and* $g_2 = \gcd(p,b)$. *Then*

$$\kappa_{r,0} = \begin{cases} (-1)^k \binom{g_1}{k} & \text{if for some } k \in \mathbb{N}\ r = \frac{p}{g_1}k, \\ 0 & \text{otherwise,} \end{cases} \tag{4.89}$$

and

$$\kappa_{0,s} = \begin{cases} (-1)^k \binom{g_1}{k} & \text{if for some } k \in \mathbb{N}\ s = \frac{p}{g_2}k, \\ 0 & \text{otherwise.} \end{cases} \tag{4.90}$$

Proof. Plug in $y = 0$ and apply the one-dimensional lemma to get the pure terms in x are

$$1 - \left(1 - x^{\frac{p}{g_1}}\right)^{g_1}. \tag{4.91}$$

Similarly, plug in $x = 0$ to get the pure terms in y. Combining these results, we obtain

$$f_{p;a,b} = 2 - \left(1 - x^{\frac{p}{g_1}}\right)^{g_1} - \left(1 - y^{\frac{p}{g_2}}\right)^{g_2} + \text{mixed terms.} \tag{4.92}$$

The formulas follow from the binomial theorem. □

Notice that Proposition 4.3.5 implies that the pure terms alternate in sign.

We now specialize to the case where $p = 2(2p+1)$, $a = 2$, and $b = 2p+1$. Using the one-dimensional case, we get

$$f_{2(2p+1);2,2p+1} = 2 - (1 - x^{2p+1})^2 - (1 - y^2)^{2p+1} + x^{2p+1} M_p(x, y) \tag{4.93}$$

where $M_p(x, y)$ is simply the mixed terms.

We compute the first few mixed terms below. In order to get a complete understanding of this case, we need to examine the polynomial sequence M_p.

$$\begin{aligned} M_0 &= 0 \\ M_1 &= 6y^2 \\ M_2 &= 20y^2 + 10y^4 \\ M_3 &= 42y^2 + 70y^4 + 14y^6 \\ M_4 &= 72y^2 + 252y^4 + 168y^6 + 18y^8 \end{aligned}$$

Proposition 4.3.6. *The polynomial sequence M_p satisfies the following linear 3rd order recurrence relation:*

$$M_{p+3} = (3 + 2y^2) M_{p+2} - (3 + y^4) M_{p+1} + (-1 + y^2)^2 M_p. \tag{4.94}$$

Using standard techniques, one can also determine the generating function. We record it here.

Corollary 4.3.7. *Let G denote the generating function of the sequence M_p. Then*

$$G = \frac{6y^2 z + z^2(2y^2 - 2y^4)}{1 - z(3 + 2y^2) + z^2(3 + y^4) - z^3(y^2 - 1)^2}. \tag{4.95}$$

Corollary 4.3.8. *The coefficients of $M_p(x, y)$ are all positive.*

In Figure 4.2, we illustrate the recurrence on the triangle of coefficients.

One could perform a similar analysis for higher values of a; however, the order of the recurrence relation will grow exponentially. Thus, using this approach to find formulas for the coefficients of $f_{p;a,b}$ in general is almost certainly hopeless.

4.3.2 An Approach to Conjecture 4.3.2

In this section we outline the strategy from [15] for proving Conjecture 4.3.2. We use the notation from [15] as much as possible while allowing for appropriate generalization. Furthermore, we attempt to point out the places where the arguments are or must be different. We also give the combinatorial interpretation of the coefficients of $f_{p;1,q}$ as the number of certain types of permutations.

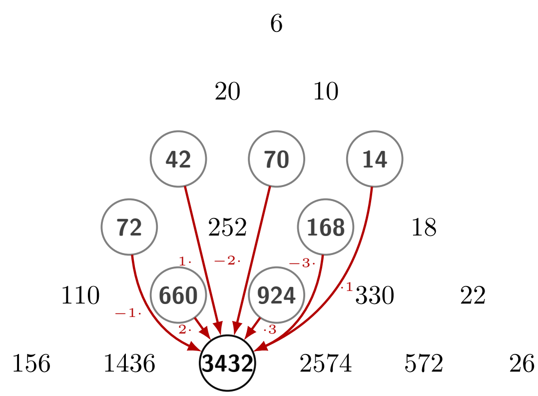

Figure 4.2: Recurrence Relation for Mixed Terms in $f_{2(2p-1);2,2p-1}$. The arrows indicate which terms are added in the recurrence relation to get the next term. The picture illustrates that $3432 = 2 \cdot 660 + 3 \cdot 924 + (-1) \cdot 72 + (-3) \cdot 168 + 42 + (-2) \cdot 70 + 14$.

Definition 4.3.9. Let a and b be positive integers, and let

$$\Theta_{p;a,b} = \prod_{j=1}^{p} \left(1 - x\omega^{aj} - y\omega^{bj}\right) = 1 - f_{p;a,b} \tag{4.96}$$

where ω is a primitive p-th root of unity.

Next we express the polynomials as circulant determinants.

Definition 4.3.10. Let $d_0, \cdots, d_{n-1} \in \mathbb{C}$. A circulant matrix is one of the form:

$$\operatorname{circ}(d_0, \cdots, d_{n-1}) = \begin{pmatrix} d_0 & d_1 & \cdots & d_{n-2} & d_{n-1} \\ d_{n-1} & d_0 & \cdots & d_{n-3} & d_{n-2} \\ \vdots & & \vdots & & \vdots \\ d_2 & d_3 & \cdots & d_0 & d_1 \\ d_1 & d_2 & \cdots & d_{n-1} & d_0 \end{pmatrix}. \tag{4.97}$$

Now we recall the stunning formula for the determinants of circulant matrices.

Proposition 4.3.11. *If ω be an n-th root of unity, then*

$$\det(\operatorname{circ}(d_0, \cdots, d_{n-1})) = \prod_{j=0}^{n-1} \left(d_0 + d_1\omega^{j} + d_2\omega^{2j} + \cdots + d_{n-1}\omega^{(n-1)j}\right) \tag{4.98}$$

.

Proof. Since

$$\begin{pmatrix} d_0 & d_1 & \cdots & d_{n-2} & d_{n-1} \\ d_{n-1} & d_0 & \cdots & d_{n-3} & d_{n-2} \\ \vdots & & \vdots & & \vdots \\ d_2 & d_3 & \cdots & d_0 & d_1 \\ d_1 & d_2 & \cdots & d_{n-1} & d_0 \end{pmatrix} \begin{pmatrix} 1 \\ \omega^j \\ \vdots \\ \omega^{(n-2)j} \\ \omega^{(n-1)j} \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \omega^j \\ \vdots \\ \omega^{(n-2)j} \\ \omega^{(n-1)j} \end{pmatrix} \tag{4.99}$$

where $\lambda = d_0 + d_1\omega^j + d_2\omega^{2j} + \cdots + d_{n-1}\omega^{(n-1)j}$. Thus λ is an eigenvalue for each j. We have found n linearly independent eigenvectors. We simply use the fact that the determinant is the product of the eigenvalues to complete the proof. □

We now write $\Theta_{p;a,b}$ as a circulant determinant.

Proposition 4.3.12. *Define*

$$d_j = \begin{cases} 1 & \textit{if } j = 0, \\ -x & \textit{if } j = a, \\ -y & \textit{if } j = b, \\ 0 & \textit{otherwise.} \end{cases} \tag{4.100}$$

Therefore the following holds:

$$\det(\mathrm{circ}(d_0, \cdots, d_{n-1})) = \Theta_{n;a,b}. \tag{4.101}$$

Proof. By the previous proposition,

$$\det(\mathrm{circ}(d_0, \cdots, d_{n-1})) = \prod_{j=0}^{n-1} \left(d_0 + d_1\omega^j + d_2\omega^{2j} + \cdots + d_{n-1}\omega^{(n-1)j} \right). \tag{4.102}$$

Plugging in $d_a = -x$, $d_b = -y$, $d_0 = 1$, and 0 for all other d_k, we get

$$\det(\mathrm{circ}(d_0, \cdots, d_{n-1})) = \prod_{j=0}^{n-1} \left(1 - x\omega^{aj} - y\omega^{bj}\right) = \Theta(n; a, b). \tag{4.103}$$

□

We pause to consider the case where $p = 6$, $a = 2$, and $b = 3$. Recall from (4.82):

$$\Theta_{6;2,3} = \prod_{j=0}^{5} \left(1 - x\omega^{2j} - y\omega^{3j}\right) \tag{4.104}$$

$$= 1 - 2x^3 + x^6 - 3y^2 - 6x^3y^2 + 3y^4 - y^6, \tag{4.105}$$

or using circulant determinants,

$$\Theta_{6;2,3} = \det \begin{pmatrix} 1 & 0 & -x & -y & 0 & 0 \\ 0 & 1 & 0 & -x & -y & 0 \\ 0 & 0 & 1 & 0 & -x & -y \\ -y & 0 & 0 & 1 & 0 & -x \\ -x & -y & 0 & 0 & 1 & 0 \\ 0 & -x & -y & 0 & 0 & 1 \end{pmatrix}. \tag{4.106}$$

The combinatorial interpretation given in [15] regards the coefficients of Θ as counting certain types of permutations.

Definition 4.3.13. Let $T_{p;a,b}(r, s)$ be the set of permutations σ such that.

1. σ has exactly $p - r - s$ fixed points,
2. for exactly r values of k, we have $\sigma(k) - k$ congruent to $a \bmod p$, and
3. for exactly s values of k, we have $\sigma(k) - k$ congruent to $b \bmod p$

Let $a(r, s)$ denote the coefficient of $x^r y^s$ in $\Theta_{p;a,b}$. We now make precise the connection between the set of permutations and the coefficient $a(r, s)$.

Proposition 4.3.14. *The following equation holds:*

$$a(r, s) = (-1)^{r+s} \sum_{\sigma \in T_{p;a,b}(r,s)} sgn(\sigma). \tag{4.107}$$

Proof. Recall the Laplace expansion for the determinant of an $p \times p$ matrix $A = (a_{ij})$:

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{p} a_{j\sigma(j)}. \tag{4.108}$$

Take A to be the circulant matrix associated to $\Theta_{p;a,b}$. Notice that $-x$ is the $a_{1,a+1}$ entry of the first row, the $a_{2,a+2}$ entry of the second row, and more generally, the $a_{j,(a+j) \bmod p}$ entry of the j-th row. Thus in order for $a_{j\sigma(j)} = -x$, we must have $\sigma(j) - j \cong a \bmod p$, since x is in the $a_{1,a+1}$ entry of the first row. Similarly, if $a_{j\sigma(j)} = -y$, we must have $\sigma(j) - j \cong b \bmod p$. The fixed points correspond to taking the identity element from the matrix. Thus $T_{p;a,b}(r,s)$ corresponds to exactly the permutations that pick out r copies of x, s copies of y, and $p-r-s$ copies of the identity element. Since x and y both appear with a negative sign, we get

$$a(r,s) = (-1)^{r+s} \sum_{\sigma \in T_{p;a,b}(r,s)} \mathrm{sgn}(\sigma). \tag{4.109}$$

□

The goal is to show that all the elements of $T_{p;a,b}(r,s)$ have the same cycle type and hence the same sign. Therefore the sign of the coefficient $a(r,s)$ is determined by r, s, and the cycle type of a single element of $T_{p;a,b}(r,s)$. Moreover, $|a(r,s)|$ is simply the cardinality of the set $T_{p;a,b}(r,s)$.

4.3.3 Examples

Recall

$$\Theta_{6;2,3} = 1 - 2x^3 + x^6 - 3y^2 - 6x^3y^2 + 3y^4 - y^6. \tag{4.110}$$

We examine elements of $T_{6;2,3}(3,2)$. We seek all permutations with exactly 6-3-2=1 fixed point, 3 increments of 2, and 2 increments of 3. For example,

$$(1\ 4\ 6\ 2\ 5)(3). \tag{4.111}$$

One can check that

$$T_{6;2,3}(3,2) = \{(1\ 4\ 6\ 2\ 5)(3), (1\ 3\ 6\ 2\ 5)(4), (1\ 3\ 6\ 2\ 4)(5), \tag{4.112}$$

$$(1\ 3\ 5\ 2\ 4)(6), (1\ 4\ 6\ 3\ 5)(2), (2\ 4\ 6\ 3\ 5)(1)\}. \tag{4.113}$$

Thus $|T_{6;2,3}(3,2)| = 6$, which agrees with $\Theta_{6;2,3}$. Also notice that all the permutations have the same cycle type, and each permutation is even, so the sign of the coefficient should be $(-1)^{3+2+2}$.

Chapter 5

Invariant Theory and CR Mappings

The goal of this chapter is to examine connections between the classical invariant theory of finite groups and the Hermitian polynomial Φ_Γ. We would like to be able to use some abstract tools to determine the signature pair of Γ. First, we give a crash course in the invariant theory of finite groups. Second, we construct Φ_Γ as a polarized version of Chern orbit polynomials. Third, we use Molien's theorem to calculate an upper bound on the total number of non-zero eigenvalues of Φ_Γ. Finally, we explore when the bound given by Molien's theorem is sharp.

5.1 Crash Course in Invariant Theory

For a more detailed account of invariant theory see [20] or [18]. Let $\pi : G \to U(n)$ be a unitary representation of a finite group G. Rather than identify G with its image as is common in most invariant theory texts, we define $\Gamma = \pi(G)$ to emphasize the connection with the representation in the notation. Let $\mathbb{C}[z_1, \cdots, z_n]$ denote the polynomial algebra in n variables over $\mathbb{C}$. We define a group action on the polynomial algebra by

$$(g \cdot h)(z_1, \cdots, z_n) = h(\pi(g^{-1})(z_1, \cdots, z_n))$$

where $h \in \mathbb{C}[z_1, \cdots, z_n]$ and $g \in G$. The set of fixed points of this action is the set of group-invariant polynomials in $\mathbb{C}[z_1, \cdots, z_n]$. Denote the set of fixed points of the action by

$$\mathbb{C}[z_1, \cdots, z_n]^\Gamma = \{h \in \mathbb{C}[z_1, \cdots, z_n] : h \circ \gamma = h \ \forall \gamma \in \Gamma\}. \tag{5.1}$$

Two of the most fundamental questions in invariant theory are:

1. Is the ring of invariant polynomials $\mathbb{C}[z_1, \cdots, z_n]^\Gamma$ finitely generated?

2. How can we construct the generators of this ring?

The first question was answered by Noether. Her result also gives us a way to find the generators.

Definition 5.1.1. The Reynolds (or averaging) operator is the map $R_G : \mathbb{C}[z_1, \cdots, z_n] \to \mathbb{C}[z_1, \cdots, z_n]^G$ given by $f \mapsto \frac{1}{|G|}\sum_{g\in G} g \cdot f$.

One can use Hilbert's basis theorem to prove that the ring of invariants is finitely generated. Furthermore, Noether proved that the ring of invariants is generated by averaging all monomials of the group.

Theorem 5.1.2. [Noether] *The ring $\mathbb{C}[z_1, \cdots, z_n]^G$ is generated by the elements $R_G(z^\alpha)$ where α is a multi-index and $|\alpha| \leq |G|$.*

Thus, the ring of invariants is finitely generated, and we can find the generators by averaging over the group all monomials of degree at most the order of the group. In order to apply these results to compute signature pairs, we are interested in the number of linearly independent elements of $\mathbb{C}[z_1, \cdots, z_n]^\Gamma$ with degree at most the order of the group Γ. Thus we are interested in $\mathbb{C}[z_1, \cdots, z_n]^\Gamma$ as a vector space instead of an algebra. Fortunately, there is a beautiful method for computing this dimension.

Define the *Molien series* by the generating function

$$\Psi(t) = \sum_{k=0}^{\infty} a_k t^k, \tag{5.2}$$

where $a_k = \dim(\mathbb{C}[z_1, \cdots, z_n]^\Gamma_k)$.

The Molien series encodes the number of linearly independent Γ-invariant polynomials in each degree. Furthermore, Molien proved that this generating function is rational. We reproduce the proof of this striking result here. See [20] for details.

Let $S_d(R_\Gamma)$ denote the matrix representing R_Γ on $\mathbb{C}[z_1, \cdots, z_n]_d$, and for $\gamma \in \Gamma$ let $S_d(\gamma)$ denote the matrix representing the action of γ on $\mathbb{C}[z_1, \cdots, z_n]_d$. Since the Reynolds operator is a projection operator, a_d equals the trace of $S_d(R_\Gamma)$. Furthermore, since the trace is additive,

$$\mathrm{tr}(S_d(R_\Gamma)) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \mathrm{tr}(S_d(\gamma)). \tag{5.3}$$

Lemma 5.1.3. *The following holds:*

$$\sum_{d=0}^{\infty} tr(S_d(\gamma))t^d = \frac{1}{\det(I_n - \gamma t)} \tag{5.4}$$

Proof. Let $w_1, \cdots, w_n$ denote the eigenvalues of γ, then $w_1, \cdots, w_n$ are the eigenvalues for $S_1(\gamma)$. Furthermore, the eigenvalues of $S_d(\gamma)$ are given by all products of d factors of $w_1, \cdots, w_n$. Since the trace is equal to the sum of its eigenvalues, we obtain

$$\operatorname{tr}(S_d(\gamma)) = \sum_{1 \leq j_1 \leq \cdots \leq j_d \leq n} w_{j_1} \cdots w_{j_d}. \tag{5.5}$$

Thus,

$$\sum_{d=0}^{\infty} \operatorname{tr}(S_d(\gamma)) t^d = \sum_{d=0}^{\infty} \sum_{1 \leq j_1 \leq \cdots \leq j_d \leq n} w_{j_1} \cdots w_{j_d} t^d = \left(\sum_{d=0}^{\infty} (w_1 t)^d \right) \cdots \left(\sum_{d=0}^{\infty} (w_n t)^d \right). \tag{5.6}$$

Use geometric series to get the following:

$$\left(\sum_{d=0}^{\infty} (w_1 t)^d \right) \cdots \left(\sum_{d=0}^{\infty} (w_n t)^d \right) = \left(\frac{1}{1 - w_1 t} \right) \cdots \left(\frac{1}{1 - w_n t} \right) = \frac{1}{\det(I_n - \gamma t)}, \tag{5.7}$$

which completes the proof. □

Theorem 5.1.4. [Molien] *For Ψ as above,*

$$\Psi(t) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{1}{\det(I_n - \gamma t)} \tag{5.8}$$

where I_n denotes the $n \times n$ identity matrix.

Proof. We sum formula (5.4) over the group and use the additivity of the trace. □

We close with an example. Let

$$\Gamma(3,2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix} \right\}. \tag{5.9}$$

Recall that

$$\mathbb{C}[z,w]^{\Gamma(3,2)} = \langle z^3, w^3, zw \rangle. \tag{5.10}$$

Then Molien's theorem gives

$$\begin{aligned}
\Psi_{\Gamma(3,2)} &= \frac{1}{3} \left(\frac{1}{(1-t)^2} + \frac{1}{(1-\omega t)(1-\omega^2 t)} + \frac{1}{(1-\omega^2 t)(1-\omega t)} \right) && (5.11) \\
&= \frac{1}{3} \left((1 + 2t + 3t^2 + 4t^3 + 5t^4 + \cdots) + 2(1 - t + t^3 - t^4 + \cdots) \right) && (5.12) \\
&= \frac{1}{3} \left(3 + 3t^2 + 6t^3 + 3t^4 + \cdots \right) && (5.13) \\
&= 1 + t^2 + 2t^3 + t^4 + \cdots . && (5.14)
\end{aligned}$$

Thus we expect to find a single invariant in degrees 0, 2 and 4, which we quickly verify: 1, zw, and z^2w^2. Similarly, we expect to find 2 degree 3 invariants, which are z^3 and w^3. Also notice that we can take the coefficients of t^2 and t^3 to get $N(\Gamma(3,2))$. We use this connection between Molien's theorem and $N(\Gamma)$ in the next section.

5.2 Molien's Theorem and an Upper Bound

We use Molien's theorem to study the total number of eigenvalues of Φ_Γ. We first introduce some notation to state the two results. The *d-jet* of a smooth function f at 0 is given by the polynomial

$$\left(J^d f\right)(t) = \sum_{k=0}^{d} \frac{f^{(k)}(0)}{k!} t^k. \tag{5.15}$$

Theorem 5.2.1. *If Γ is a finite subgroup of $U(n)$ with order p, then*

$$N(\Gamma) + 1 \leq J^p \left(\frac{1}{p} \sum_{\gamma \in \Gamma} \frac{1}{\det(I_n - t\gamma)} \right) \Bigg|_{t=1}. \tag{5.16}$$

Proof. Since $\Phi_\Gamma(0) = 0$, the invariant 1 is not included in $N(\Gamma)$ (that is why the plus 1 term is on the left-hand-side). Otherwise, each eigenvalue must have an associated invariant polynomial. Since we only count linearly independent polynomials, we obtain

$$N(\Gamma) + 1 \leq \sum_{d=0}^{|\Gamma|} a_d. \tag{5.17}$$

Rewriting the right-hand-side in terms of jets completes the proof. □

Equality holds in (5.16) for cyclic subgroups of $U(2)$.

Corollary 5.2.2. *If Γ is a cyclic subgroup of order p in $U(2)$, then*

$$N(\Gamma) + 1 = J^p \left(\frac{1}{p} \sum_{\gamma \in \Gamma} \frac{1}{\det(I_2 - t\gamma)} \right) \Bigg|_{t=1}. \tag{5.18}$$

Notice that for every subgroup of $U(2)$ considered here, equality holds. That leads us to the following conjecture.

Conjecture 5.2.3. *If Γ is a finite subgroup of order p in $U(2)$, then*

$$N(\Gamma) + 1 = J^p \left(\frac{1}{p} \sum_{\gamma \in \Gamma} \frac{1}{\det(I_2 - t\gamma)} \right) \Bigg|_{t=1} . \tag{5.19}$$

One may be tempted to conjecture that this equality holds for all finite subgroups of $U(n)$, but it is not the case. We present a counterexample now. Consider the cyclic group $\Gamma(6; 1, 2, 4)$ generated by

$$\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^4 \end{pmatrix} \tag{5.20}$$

where ω is a 6-th root of unity. Then,

$$\begin{aligned} f_{6;1,2,4} = {} & x^6 + 6x^4y + 9x^2y^2 + 2y^3 - y^6 + 6x^2z + 6yz - 6x^2y^3z - 6y^4z - 3x^4z^2 - 9y^2z^2 \\ & + 2z^3 - 2y^3z^3 + 3x^2z^4 - 6yz^4 - z^6. \end{aligned} \tag{5.21}$$

Thus $N(\Gamma) = 16$. However,

$$J^6 \left(\frac{1}{6} \sum_{\gamma \in \Gamma} \frac{1}{\det(I_2 - t\gamma)} \right) \Bigg|_{t=1} - 1 = 17. \tag{5.22}$$

The invariant term x^2yz^2 doesn't appear in Φ_Γ, so equality does not always hold.

Conjecture 5.2.4. *If Γ is a cyclic subgroup of order p in $U(2)$ and p is prime, then*

$$N(\Gamma) + 1 = J^p \left(\frac{1}{p} \sum_{\gamma \in \Gamma} \frac{1}{\det(I_2 - t\gamma)} \right) \Bigg|_{t=1} . \tag{5.23}$$

It may be possible to prove this conjecture using the techniques of Loehr, Warrington, and Wilf. Another possible direction for future research is to find an analogue to Molien's theorem that encodes both the numbers of positive and negative eigenvalues of Φ_Γ simultaneously.

5.3 Orbit Polynomial and Chern Orbit Classes

In this section we describe how the group-invariant Hermitian polynomials Φ_Γ arise in the context of representation theory. Theorem 1.1.5 will express Φ_Γ in terms of the orbit Chern classes. We begin by recalling some basic definitions. In particular, we define the orbit polynomial and orbit Chern classes as in [18].

Define $G \cdot h$ to be the G-orbit corresponding to $h \in \mathbb{C}[z_1, \cdots, z_n]$. Following [18], define the orbit polynomial of $G \cdot h$ by

$$\phi_{G\cdot h}(X) = \prod_{b \in G\cdot h} (X + b). \tag{5.24}$$

Expanding the product we get

$$\phi_{G\cdot h}(X) = \sum_{a+b=|G|} c_a(G \cdot h) X^b \tag{5.25}$$

where $c_a(G \cdot h) \in \mathbb{C}[z_1, \cdots, z_n]^G$ are called the orbit Chern classes of the orbit $G \cdot h$. The definition of orbit Chern class agrees with the usual topological definition of Chern class; this construction is given in [19].

Now we restrict our attention to the case $n = 2$, G is a cyclic group of order p, and the representation is given by $\pi(G) = \Gamma(p, q)$. Let $h = -(z_1 + z_2)$, then $G \cdot h = \left\{-\omega^j z_1 - \omega^{qj} z_2 : j = 0, \cdots p-1\right\}$. The orbit polynomial $\phi_{G\cdot h}(X) = \prod_{j=0}^{p-1} \left(X - \omega^j z_1 - \omega^{qj} z_2\right)$. Thus the orbit polynomial evaluated at 1 is $f_{p,q}$. Taking the total Chern class of the orbit $G \cdot h$ gives exactly $f_{p,q}$.

We can polarize Φ_Γ; we treat z and $\overline{z}$ as independent variables. Thus we write

$$\Phi_\Gamma(z, \overline{w}) = 1 - \prod_{\gamma \in \Gamma} (1 - \langle \gamma z, w \rangle). \tag{5.26}$$

If we set $w = (1, 1, \cdots, 1)$, then $\Phi_\Gamma(z, \overline{w}) = 1 - \prod_{\gamma\in\Gamma} \left(1 - \sum_{j=1}^n \gamma z_j \overline{w_j}\right)$, which is exactly the alternating sum of orbit Chern classes of the orbit corresponding to $z_1 + z_2 + \cdots + z_n$. We obtain the following theorem and corollary.

Theorem 5.3.1. *Let $\pi : G \to U(n)$ be a faithful, unitary representation of the finite group G. Put $\Gamma = \pi(G)$. Then*

$$\Phi_\Gamma(z, 1) = \sum_{j=1}^{p} (-1)^{j-1} c_j(G \cdot (z_1 + \cdots + z_n)). \tag{5.27}$$

Corollary 5.3.2. *In Theorem 5.3.1, put $\pi(G) = \Gamma(p, q)$. Then*

$$f_{p,q}(x, y) = \sum_{j=1}^{p} (-1)^{j-1} c_j(G \cdot (x + y)). \tag{5.28}$$

Appendix A

Mathematica Source Code for Computing Signature Pairs

Let Γ be a finite subgroup of $U(2)$, and recall that

$$\Phi_\Gamma(z, \bar{w}) = 1 - \prod_{\gamma \in \Gamma} (1 - \langle \gamma z, w \rangle).$$

In order to compute the eigenvalues of Φ_Γ, we use the Mathematica [13] code below. The function GroupSignaturePair uses standard Mathematica commands to compute the eigenvalues. The function takes a list of the group elements of Γ and returns a list of the eigenvalues of the underlying Hermitian matrix of coefficients of the polynomial Φ_Γ. Computing signature pairs in this way is very memory intensive. To improve performance, one can use Mathematica commands to numerically find the eigenvalues.

```
GroupSignaturePair[group_] :=
Module[{matrix, hermitianmatrix, poly, order, eigenvalues},
  poly = First[Expand[1 - Product[1 - Transpose[(L.{{z1}, {z2}})].{w1, w2}, {L, group}]]];
  order = Length[group];
  matrix = CoefficientList[poly, {z1, z2, w1, w2}];
  hermitianmatrix = Partition[Flatten[matrix], (order + 1)^2];
  eigenvalues = Eigenvalues[hermitianmatrix];
  Return[eigenvalues]]
```

References

[1] J. P. D'Angelo, *Proper polynomial mappings between balls*, Duke Math J. **57** (1988), 211 –219.

[2] ———, *Several complex variables and the geometry of real hypersurfaces*, CRC Press, Boca Raton, FL, 1992.

[3] ———, *Invariant holomorphic mappings*, J. of Geometric Analysis **6** (1996), no. 2, 163 –179.

[4] ———, *Number-theoretic properties of certain CR mappings*, J. of Geometric Analysis **14** (2004), no. 2, 215 –229.

[5] ———, *Asymptotics for invariant polynomials.*, Complex Var. Elliptic Equ. **52** (2007), no. 4, 261 –272.

[6] J. P. D'Angelo, S. Kos, and E. Riehl, *A sharp bound for the degree of proper monomial mappings between balls*, J. of Geometric Analysis **13** (2003), no. 4, 581 –593.

[7] J. P. D'Angelo and D. Lichtblau, *Spherical space forms, CR mappings, and proper maps between balls.*, J. Geom. Anal. **2** (1992), no. 5, 391 –415.

[8] K. Dilcher and K. Stolarsky, *A pascal-type triangle characterizing twin primes.*, Amer. Math. Monthly **112** (2005), no. 8, 673 –681.

[9] P. Du Val, *Homographies, quaternions and rotations*, Oxford Mathematical Monographs Clarendon Press, Oxford, 1964.

[10] F. Forstnerič, *Extending proper holomorphic mappings of positive codimension.*, Invent. Math. **95** (1989), no. 1, 31 –61.

[11] N. Garnier and O. Ramaré, *Fibonacci numbers and trigonometric identities*, Fibonacci Quart. **47** (2009), no. 1, 56 –61.

[12] D. Grundmeier, *Signature pairs for group-invariant Hermitian polynomials*, International J. of Math. **22** (2011), no. 3, 311 –343.

[13] Wolfram Research Inc., *Mathematica edition: Version 7.0*, Wolfram Research, Inc., Champaign, Illinois, 2008.

[14] D. Lichtblau, *Invariant proper holomorphic maps between balls.*, Indiana Univ. Math. J. **41** (1992), 213 –231.

[15] N. Loehr, G. Warrington, and H. Wilf, *The combinatorics of a three-line circulant determinant.*, Israel J. Math. **143** (2004), 141 –156.

[16] G. Musiker, *Combinatorial aspects of elliptic curves.*, Sém. Lothar. Combin. **56** (2006).

[17] T. Osler, *Cardan polynomials and the reduction of radicals*, Mathematics Magazine **74** (2001), no. 1, 26 –32.

[18] L. Smith, *Polynomial invariants of finite groups*, A K Peters, Wellesley, MA, 1995.

[19] L. Smith and R. E. Stong, *On the invariant theory of finite groups: orbit polynomials and splitting principles.*, J. Algebra **110** (1987), 134 –157.

[20] T. A. Springer, *Invariant theory*, Springer, New York, NY, 1977.